16. Remarks on Nonlinearity and Chaos

16.1 Determinism vs Predictability

In Newtonian mechanics, dynamical systems are described by differential equations which, if supplied with initial conditions, can be integrated to a unique solution at any future time \( t \). A unique solution means that for each set of initial conditions there exists only one solution. A set of initial conditions determines exactly one solution. Dynamical systems having this property are called deterministic. See Example 12.8.

In simple systems such as occur in the examples and problems of the preceding chapters, the solution can be written explicitly in terms of elementary functions containing the initial conditions as parameters. The correspondence between the initial conditions and solutions is then quite transparent. One may come to suspect that the correspondence is always this 'regular', in particular that two sets of initial conditions that are almost identical will also correspond to two solutions that are almost identical for all times.

This, however, is not true in general. Many, even conceptually simple, dynamical systems have the property that their solutions depend in an extremely sensitive manner on the initial conditions, in such a way that nearly identical initial conditions quickly evolve to completely different solutions. The number of digits needed to specify an initial condition in order to make a meaningful prediction a time \( t \) ahead, can be an exponentially increasing
function of \( t \). If, say, one additional decimal place is required in the specification of the initial values for each second ahead that one wishes to predict, the exact solution quickly becomes impossible to predict from an integration of the equations of motion. Dynamical systems with this property are loosely termed 'chaotic', and have recently (not least spurred by the evolution of inexpensive, high-precision, high-speed computers) been the focus of much attention.

A complete understanding and classification of all such systems seems beyond the abilities of present-day mathematics, even though much progress has been made. The problems involved have forced mathematicians and physicists to re-evaluate the very concept of what we mean by the 'solution' to a dynamical system. In the study of such 'chaotic' systems, the emphasis has shifted towards obtaining qualitative, global information about the system, rather than seeking the type of detailed and local solutions discussed in the preceding chapters.

The mathematical techniques involved in gaining even a qualitative understanding of chaotic systems are fairly sophisticated. In this chapter we shall merely try to give the reader a glimpse of some of the aspects of the topic.

It is essential, however, to be aware of the following fact: even the most 'chaotic' macroscopic system obeys all the fundamental laws of Newtonian mechanics, summarized in Chapter 12. Recent advances in mathematical and computational techniques have expanded our insight into the many complex types of behavior that solutions to the equations of motion may have. But the principles and conservation laws that we use to establish the differential equations are unchanged. Mechanical systems remain governed by laws and concepts laid down by Isaac Newton more than 300 years ago, and extended by Albert Einstein early in the 20th century.

### 16.2 Linear and Nonlinear Differential Equations

The mathematical description of mechanical systems that you have seen in the preceding chapters, has been in the form of ordinary differential equations. In mathematics courses you will learn about the many interesting properties of this class of differential equations. It is important to distinguish between linear and nonlinear differential equations. An ordinary differential equation in the unknown function \( x(t) \) is said to be **linear**, if the differential equation only contains terms linear in \( x(t) \) and its derivatives, i.e., if it can be written in the following form:

\[
a_n(t) \frac{d^n x}{dt^n} + \cdots + a_1(t) \frac{dx}{dt} + a_0(t)x + f(t) = 0 ,
\]

where the functions \( a_i(t) \) and the function \( f(t) \) are arbitrary. Ordinary differential equations that are not linear, are called **nonlinear**. One can subse-