Chapter 11. Conformal Mapping of a Circle

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11.1 Introduction

Mapping techniques are mathematical methods which are frequently applied for solving fluid flow problems in the interior involving bodies of nonregular shape. Since the advent of supercomputers such techniques have become quite important in the context of numerical grid generation [1]. In introductory courses in fluid dynamics students learn how to calculate the circulation of an incompressible potential flow about a so-called “Joukowski airfoil” [3] which represent the simplest airfoils of any technical relevance. The physical plane where flow about the airfoil takes place is in a complex $p = u + iv$ plane where $i = \sqrt{-1}$. The advantage of a Joukowski transform consists in providing a conformal mapping of the $p$-plane on a $z = x + iy$ plane such that calculating the flow about the airfoil gets reduced to the much simpler problem of calculating the flow about a displaced circular cylinder. A special form of the mapping function $p = f(z) = u(z) + iv(z)$ of the Joukowski transform reads

$$p = \frac{1}{2}(z + \frac{a^2}{z}). \quad (11.1)$$

In this chapter we shall demonstrate how the mathematical transformations required in applying mapping methods can be handled elegantly by means of a language for symbolic computation and computer algebra. Rather than choosing a large physical problem that would be beyond the scope of this book, we select a very simple application of conformal mapping to illustrate the essential steps involved.

11.2 Problem Outline

It is suitable to express both $x$ and $y$ in terms of some parameter $t$ such that $z(t) = x(t) + iy(t)$. Consequently also $p(z(t)) = f(z(t))$ holds. Inserting expression $z(t)$ into $p(z) = f(z(t))$ leads to

$$P(t) = U(t) + iV(t) = p(z(t)) \quad (11.2)$$

Twofold differentiation on either side of the equal sign with respect to $t$ yields at next

$$\dot{P}(t) = p'(z(t))\dot{z}(t), \quad (11.3)$$
and subsequently

\[ \ddot{P}(t) = p''(z(t))(\dot{z}(t))^2 + p'(z(t))\ddot{z}(t) \]  

(11.4)

where dots and primes denote derivatives with respect to \( t \) and \( z \), respectively.

The relations outlined so far are needed for handling the following problem. We assume a mapping function \( p \) which results from solving the second order differential equation presented in [2]

\[ z^2p'' + zp' + (\alpha z^2 + \beta z + \gamma)p = 0 \]  

(11.5)

subject to the initial conditions \( p'(0) = p'_0 \) and \( p(0) = p_0 \).

Obviously the problem of mapping \( z(t) \) on the domain \( Y(t) = U(t) + iV(t) \) requires the solution of the complex second order differential equation specified by (11.5). This can be done by proceeding as follows: For the considered problem \( z(t) \) is chosen as a circle of radius \( r \), i.e. \( z = re^{it} \). Hence, derivatives of \( z(t) \) with respect to \( t \) can be readily evaluated. Inserting the complex expressions for \( z(t), \dot{z}(t), \) and \( \ddot{z}(t) \), respectively, into Equations (11.2)-(11.5) permits one to completely eliminate any explicit appearance of \( z \) and its \( t \)-derivatives from all these equations. After this \( p' \) and \( p'' \) in (11.3) and (11.4) can be expressed only in terms of \( \dot{P}(t) \) and \( \ddot{P}(t) \). Finally, when using these expressions together with (11.2), our second order differential equation can be modified such that it only entails \( P(t), \dot{P}(t), \) and \( \ddot{P}(t) \). Considering (11.2) we can simplify this complex differential equation by collecting the real and imaginary components. This yields a system of two coupled second order differential equations. The solution \( U(t) \) and \( V(t) \) can now be obtained by application of a standard numerical integration algorithm.

For the following two sets of initial conditions and values of parameter \( r \) illustrative solutions can be found:

\[
\begin{align*}
  r & = 1 & r & = 0.6 \\
  U(0) & = -0.563 & U(0) & = -0.944 \\
  \dot{U}(t) & = 0 & \dot{U}(t) & = 0 \\
  V(0) & = 0 & V(0) & = 0 \\
  \dot{V}(t) & = 0.869 & \dot{V}(t) & = 0.658 
\end{align*}
\]

In each of the two runs integration is done over the interval \( 0 \leq t \leq 6\pi \). Moreover, the following parameter values will be used \( \alpha = 1, \beta = 0.5, \) and \( \gamma = -4/9 \). Subsequently the latter three parameters will be referred to as \( a, b, \) and \( c \).

### 11.3 Maple Solution

The solution steps outlined above are not very difficult to perform. Anyhow, since there are several manipulations required, deriving the formulas manually and programming them in any of the well known higher computer languages