Chapter 4. Orbits in the Planar Three-Body Problem

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4.1 Introduction

The planar three-body problem is the problem of describing the motion of three point masses in the plane under their mutual Newtonian gravitation. It is a popular application of numerical integration of systems of ordinary differential equations since most solutions are too complex to be described in terms of known functions.

In addition, the three-body problem is a classical problem with a long history and many applications (see e.g. the exhaustive accounts by Szebehely [9] or by Marchal [4]). Nevertheless, the sometimes complicated interplay of the three bodies can often be described in terms of two-body interactions and is therefore qualitatively simple to understand. About 100 years ago the French Academy of Sciences set out a prize for the solution of the problem which was awarded to Sundman [7] for a series solution convergent at all times. However, owing to the excessively slow convergence of Sundman's series it is of no practical value for discussing orbits.

In this article we will demonstrate how MAPLE and MATLAB can be used efficiently to construct and display numerical solutions of the planar three-body problem. In Section 4.2 we will straightforwardly use the differential equations of motion and the numerical integrator of MATLAB. Although for most initial conditions this approach will quickly produce an initial segment of the solution, it will usually fail at a sufficiently close encounter of two bodies, owing to the singularity at the corresponding collision.

In classical celestial mechanics the regularizing transformation by T. Levi-Civita [3] is an efficient technique to overcome the problems of numerically integrating over a collision or near-collision between two bodies. Since three different pairs can be formed with three bodies it was suggested by Szebehely and Peters [8] to apply Levi-Civita's transformation to the closest pair if the mutual distance becomes smaller than a certain limit.

In Section 4.3 we will use a set of variables suggested by Waldvogel [10] that amounts to automatically regularizing each of the three types of close encounters whenever they occur. Owing to the complexity of the transformed equations of motion, the Hamiltonian formalism will be used for deriving these
equations. Then MAPLE’s capability of differentiating algorithms (automatic differentiation) will be used to generate the regularized equations of motion.

4.2 Equations of Motion in Physical Coordinates

Let $m_j > 0$ ($j = 0, 1, 2$) be the masses of the three bodies, and let $x_j \in \mathbb{R}^2$ and $\dot{x}_j \in \mathbb{R}^2$ be their position and velocity (column) vectors in an inertial coordinate system (dots denoting derivatives with respect to time $t$). For the mutual distances of the bodies the notation of Figure 4.1 will be used:

$$r_0 = |x_2 - x_1|, \quad r_1 = |x_0 - x_2|, \quad r_2 = |x_1 - x_0|.$$  

(4.1)

**Figure 4.1. The Three-Body Problem in Physical Coordinates.**

Next we notice that the Newtonian gravitational force $F$ exerted onto $m_0$ by $m_1$ is given by

$$F = -m_0 m_1 \frac{x_0 - x_1}{|x_0 - x_1|^3}$$

if the units of length, time and mass are chosen such that the gravitational constant has the value 1. Therefore the Newtonian equations of motion (in their most primitive form) become

$$\ddot{x}_0 = m_1 \frac{x_1 - x_0}{r_2^3} + m_2 \frac{x_2 - x_0}{r_1^3}$$
$$\ddot{x}_1 = m_2 \frac{x_2 - x_1}{r_0^3} + m_0 \frac{x_0 - x_1}{r_2^3}$$
$$\ddot{x}_2 = m_0 \frac{x_0 - x_2}{r_1^3} + m_1 \frac{x_1 - x_2}{r_0^3}.$$  

(4.2)