Chapter 7. The Generalized Billiard Problem

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7.1 Introduction

We consider the following problem: Given a billiard table (not necessarily rectangular) and two balls on it, from which direction should the first ball be struck, so that it rebounds off the rim of the table, and then impacts the second ball? This problem has previously been solved for a circular table in [1, 3].

Let the billiard table be parametrically described by \( X = f(t) \) and \( Y = g(t) \). These functions may be arbitrary, with the single requirement being that their first derivatives exist. We aim at the rim points described by the coordinates \([X(t_i), Y(t_i)]\). To solve the given problem, we shall use two different methods; the generalized reflection method and the shortest trajectory method. We use both methods to derive functions of the parameter \( t \), whose roots are the points \( t_1, \ldots, t_n \); i.e. the solution to the given problem. We shall first find the analytic solution in the first part of the chapter. In the second part we intuitively check this solution by solving some practical examples. In more complicated cases we shall use numerical methods to solve the final equation.

7.2 The Generalized Reflection Method

The solution of the problem can be divided into two main steps. First, the trajectory of the ball reflected by the general point on the billiard rim is found. Second, the point on the billiard rim is found such that the second ball lies on this trajectory.

The first step can be solved by a generalization of the plane mirror problem, i.e. find the path touching line \( l \) from point \( P \) to point \( Q \), according to the mirror condition. The mirror condition is satisfied when the impact angle is equal to the reflection angle.

In the second step we calculate the distance between the reflected trajectory and the position of the second ball. This distance is a function of the impact position point. The problem is solved once the point has been found with corresponding distance equal to zero.
7.2.1 Line and Curve Reflection

To begin, we construct point $M$ as a mirror point of $P$, using the line $l$ as an axis of symmetry. The line $l'$ connecting points $M$ and $Q$ intersects line $l$ at point $T$. Now, the required path passes through point $T$ as shown in Figure 7.1). From point $P$ one must aim at point $T$ in order to hit the ball located at point $Q$. As the mirror line we use a tangent line of the table boundary. It is possible to describe this line implicitly:

\[ l \equiv l_1 \equiv y - T_y = k(x - T_x), \quad k_1 \equiv k = \frac{dX(t)}{dY(t)} \mid_T \]

(7.1)

where

\[ T \equiv [T_x, T_y] = [X(t), Y(t)], \]

and where $T_x, T_y$ are the coordinates of the common point on the billiard table boundary and $k$ is the derivative of the boundary function at this point. The ball hitting the boundary is reflected just as if it were reflected by this line. To find the mirror image of the position of one ball we use $l$ as a mirror line. The solution of the generalized billiard problem is shown in Figure 7.2. We can aim the ball located at $P$ toward a point $T$ on the billiard boundary. We use the line $l_1$, tangent to the boundary at point $T$, to find point $M$, the mirror image of point $P$. $l_2$ is the line segment connecting points $P$ and $M$. $l_1$ is perpendicular to $l_2$, intersecting $l_2$ in its midpoint $C$.

If point $M$ is known, then it is possible to draw the line segment $l_3$ by connecting points $M$ and $Q$, and the ray $l_4$ (beginning at point $M$ and going through point $T$). The ball located at point $P$ will move along this ray after its reflection. Now we can calculate the distance $d_2$ between point $Q$ and ray $l_4$. If this distance is equal to zero, i.e., $d_2 = 0$, then the position of the second ball lies on the trajectory of the first ball after its reflection. Hence, the balls must collide. A second possibility is to calculate the distance $d_1$ between points $T$ and $V$, the intersection of lines $l_1$ and $l_3$. To hit the ball located at point $Q$, this distance must also be equal to zero, i.e. $d_1 = 0$, since in this case $T$ and $V$ must coincide; i.e., $l_4$ will then coincide with $l_3$. The calculation of the distance $d_1$ is simpler than the calculation of $d_2$. To determine the distance between two