Singularity Theory for Nonlinear Optimization Problems

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1. Background

Consider a smooth (C^∞) function $f : R^n \to R^m$ and assume that f has a critical point at the origin, i.e., $df(0) = 0$. The theory of singularities as developed by Thom, Mather, Arnol'd, and others (Lu, 1976; Gibson, 1979; Arnol'd, 1981) addresses the following basic questions:

(1) What is the local character of f in a neighborhood of the critical point? Basically, this question amounts to asking "at what point is it safe to truncate the Taylor series for f?" This is the *determinacy* problem.
(2) What are the "essential" perturbations of f? That is, what perturbations of f can occur that change the qualitative nature of f and that cannot be transformed away by a change of coordinates? This is the *unfolding* problem.
(3) Can we classify the types of singularities that f can have up to diffeomorphism? This is the *classification* problem.

Elementary catastrophe theory largely solves these three problems (when $m = 1$); its generalization to singularity theory solves the first two, and gives relatively complete information on the third for m, n small. Here we outline a program for the utilization of these results in an applied setting to deal with certain types of nonlinear optimization problems. In the following section we give a brief summary of the main results of singularity theory for problems (1)–(3) for *functions* ($m = 1$) and then proceed to a discussion of how these results may be employed for nonlinear optimization.

2. Determinacy, Unfoldings, and Classifications

2.1. Equivalence of germs

In its local version, elementary catastrophe theory deals with functions $f : U \to R$ where U is a neighborhood of O in R^n. The cleanest way to handle

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such functions is to pass to *germs*, a germ being a class of functions that agrees on suitable neighborhoods of O. All operations on germs are defined by performing similar operations on representatives of their classes. In the sequel, we usually make no distinction between a germ and a representative function.

We let E_n be the set of all smooth germs $R^n \to R$, and let E_{nm} be the set of all smooth germs $R^n \to R^m$. Of course $E_{n,1} = E_n$. These sets are vector spaces over R, of infinite dimension. We abbreviate $(x_1, \ldots, x_n) \in R^n$ to x. If $f \in E_{nm}$ then

$$f(x) = [f_1(x), \ldots, f_m(x)]$$

and the f_i are the *components* of f.

A *diffeomorphism* germ $\varphi : R^n \to R^n$ satisfies $\varphi(0) = 0$, and has an inverse φ' such that $\varphi[\varphi'(x)] = x = \varphi'[\varphi(x)]$ for x near 0. It represents a smooth, invertible local coordinate change. By the Inverse Function Theorem, φ is a diffeomorphism germ if and only if it has a nonzero Jacobian, that is,

$$\det[\partial\varphi_i / \partial x_j(0)] \neq 0$$

Two germs $f, g : R^n \to R$ are *right equivalent* if there is a diffeomorphism germ φ and a constant $\gamma \in R$ such that

$$g(x) = f\,\varphi(x) + \gamma$$

This is the natural equivalence for studying topological properties of the gradient ∇f (Poston and Stewart, 1978). If f, rather than ∇f, is important, the term γ is omitted.

A *type* of germ is a right equivalence class and the classification of germs up to right equivalence amounts to a classification of types. Each type forms a subset of E_n, and the central object of study is the way these types fit together.

A precise description is complicated by the fact that most types have infinite dimension; but there is a measure of the complexity of a type, the *codimension*, which is generally finite. Heuristically, it is the difference between the dimension of the type and that of E_n (even though both are infinite). A precise definition is given below.

The largest types have codimension 0 and form open sets in E_n. Their boundaries contain types of codimension 1; the boundaries of these in turn contain types of codimension 2, and so on, with higher codimensions revealing progressively more complex types. Types of infinite codimension exist, but form a very small set in a reasonable sense.