When these solutions are introduced in eqs. (111) and (112d, e), it is found that all stress resultants may be brought into the form:

\[ f = c y e^{\gamma n y} \left[ (a_1 \tilde{C}_1 + a_2 \tilde{C}_2) \cos (\mu_1 \ln y) + (a_1 \tilde{C}_2 - a_2 \tilde{C}_1) \sin (\mu_1 \ln y) \right] + y^{-n} \left[ (a_3 \tilde{C}_3 + a_4 \tilde{C}_4) \cos (\mu_1 \ln y) + (a_3 \tilde{C}_4 - a_4 \tilde{C}_3) \sin (\mu_1 \ln y) \right] \cos n \theta. \] (121)

The values of \( c, e, \) and the expressions for \( a_1 \) and \( a_2 \) are given in Table 9.

To obtain expressions for \( a_3 \) and \( a_4 \), one has to replace \( \tilde{x}_1, \tilde{x}_2 \) by \( \tilde{x}_3, \tilde{x}_4 \); \( \tilde{\beta}_1, \tilde{\beta}_2 \) by \( \tilde{\beta}_3, \tilde{\beta}_4 \); and \( \kappa_1 \) by \(- \kappa_1\), while \( \mu_1 \) remains unaltered. When there is another set of four complex roots, they are dealt with in the same way. When there are some complex and some real roots, each part of the solution must be handled in its own way, but when they are all superposed there are always 8 real constants of integration available, whether they are real constants \( C_i \) or the real and imaginary parts \( \tilde{C}_i \) of complex constants.

Chapter 7.

BUCKLING OF SHELLS

7.1 Introduction

In many examples in the preceding chapters we have seen that shells can be very thin-walled and that they very often are subjected to compressive stresses in extensive areas. The question arises whether the elastic equilibrium of such shells is stable. To answer this question, one of the standard methods of the theory of elastic stability must be applied: the method of adjacent equilibrium or the energy method. We shall explain here the basic ideas of both methods in the terminology of shells and then consider an Euler column to demonstrate their use simply.

7.1.1 Adjacent Equilibrium

We consider a shell carrying a certain load, which we shall call the basic load. It produces the basic stresses and the basic displacements.

We disturb the elastic equilibrium by imposing a small additional deformation, say, some lateral deflection. Every such deformation is connected with strains and hence with stresses, and we may expect that certain external forces will be needed to produce it. When these forces are removed, the whole disturbance vanishes. If this situation prevails, the elastic equilibrium is stable.

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When the basic load is increased, it may happen that less force is needed to produce the same disturbance and that, at last, a certain disturbance becomes possible without any disturbing forces. In such a case the elastic equilibrium is neutral with respect to this particular disturbance. It may be shown quite generally\(^1\) that the elastic equilibrium is always stable when the basic load is small enough. The smallest value which the load must assume to reach a neutral equilibrium is called the critical load or buckling load. The disturbance, i.e. a system of additional stresses and displacements, may then occur spontaneously, and this phenomenon is the buckling of the shell.

When the basic load is increased beyond its critical limit, the elastic equilibrium becomes unstable, and any incidental disturbance causes the shell to leave entirely its initial position of equilibrium. Whether or not this leads to a collapse is a question still to be discussed (see the papers on post-buckling behavior mentioned in the bibliography).

Actually to find the buckling load, we proceed in this way: We formulate the differential equations for the disturbed equilibrium without a disturbing load and ask whether these equations, together with appropriate boundary conditions, admit a solution. These equations contain, of course, all the terms which occur in the equations for the undisturbed equilibrium. They also contain terms with the additional stresses (or stress resultants). Since the disturbance is supposed to be very small (infinitesimal, if we wish), these new terms are very small, and since they are essential for our problem, we must take all terms of the same order of magnitude. There is a second group of such terms resulting from the fact that the basic load is now acting on a slightly deformed element. As we shall see later in more detail, these terms consist of products of a basic force or stress resultant with an additional displacement or its derivative.

Both groups of small terms are proportional to the disturbance: the first to the stress resultants and the others to the displacements which are added to the basic state. Since the conditions of equilibrium are satisfied without all these terms (i.e. for the undisturbed case), the small terms by themselves must add up to zero in every equation. And since Hooke's law expresses the stress resultants in terms of the displacements, we arrive at last at a set of homogeneous linear differential equations for these displacements \(u, v, w\).

Now let us look at the boundary conditions to which these buckling displacements are subjected. Whatever conditions we impose on the basic state, the same conditions will be imposed on the buckled state. When we subtract the one from the other, we see that the buckling