Chapter 6

New Developments in the Theory of Dynamical Systems

The models described in the preceding chapters have been analyzed by means of mathematical techniques which essentially belong to the nowadays standard and basic knowledge in dynamical systems theory. While the Poincaré-Bendixson theorem, the special features of the van-der-Pol equation or the Liénard equation etc. have been used especially in engineering problems for a fairly long time, it is only the introduction of these concepts into economics which is relatively new. In this sense, it may be appropriate to label the techniques used in Chapter 5 as classical methods.

While during the first half of the century oscillation phenomena received mainly the attention of engineers only - especially in electric circuit devices -, a tremendous effort has been made by mathematicians and scientists since the mid-Sixties to gain new insights into dynamical systems beyond the classically known features and to uncover new fields of application of dynamical systems theory in different branches of science. Forced by the intensive and steadily increasing usage of fast large-scale computers, it had been discovered that a large variety of dynamic phenomena exists in addition to those which could be dealt with classical techniques. Furthermore, practical problems in different fields made it evident that the mathematical dynamical systems theory was rather unsatisfactory. Problems such as phase transitions in laser devices in physics, the morphogenesis in biology and chemistry, thermodynamic phenomena far from equilibrium in chemistry, or turbulences in physics and meteorology could not be analyzed by means of standard methods. Inspired by E.N. Lorenz's pioneering work on turbulences in meteorology and R. Thom's work on morphogenesis, a revival of e.g. the mathematical field of differential topology was initiated which by no means can be considered
satisfactorily completed. Despite this open character of modern dynamical systems theory whose scientific route in the future is unclear, the outcomes of the theory such as the intensification of bifurcation theory, the catastrophe theory, synergetics, or chaos theory can also be viewed as having a potentially substantial influence on the analysis of economic dynamics and business cycle theory. Although in many cases it is not possible to apply the techniques which were introduced e.g. in physics directly also to economics, these new methods at least show how restrictive the concentration on the usual methods in common business cycle theory has been in the face of the large variety of possible complex behaviors.

As the mathematical apparatus of these new developments encloses a huge number of definitions, theorems, and concepts, it would be far beyond the scope of this book and beyond the capabilities of the authors to provide a satisfactory mathematical overview. Rather, it is attempted to illustrate the basic ideas and to stress those features which are important for an economist concentrating on the technical usefulness of a mathematical theory in economic analysis.

6.1. Dynamical Systems and Transitions

The central expression of this chapter is bifurcation. In general, the term bifurcation describes the occurrence of a qualitative change in the solution of a dynamical system. Specifically, this section is concerned with bifurcations that arise from changing of exogenously given parameters, i.e. with situations in which a stable fixed point splits into two fixed points or into a limit cycle, for instance. To make some ideas more precise, consider the differential equation system

\[ \dot{x} = f(x, \mu), \quad x \in A \subset \mathbb{R}^n, \quad \mu \in \mathbb{R} \]  

(6.1.1)

in which \( x \) denotes the vector of the state variables and \( \mu \) is an exogenously given parameter. Let \( x(t, x_0, \mu) \) be a solution of (6.1.1). Two solutions \( x(t, x_0, \mu_1) \) and \( x(t, x_0, \mu_2) \) for different values of the control parameter \( \mu \) are said to be equivalent if the corresponding trajectories have the same topological structure, e.g. if for small differences in \( \mu \) both trajectories form closed orbits which are close together. If two solutions are equivalent, the system (6.1.1) is said to be generic or structurally stable.

There may be systems like (6.1.1), which are structurally stable only locally for certain intervals of the parameter \( \mu \). Let \( S \) be the subset \( S \subset \mathbb{R} \) of all \( \mu \), for which the system (6.1.1) is structurally stable. The complementary set \( B = \mathbb{R} \setminus S \) is called the set of bifurcation points.

---

1 See Cugno/Montrucchio (1984) for an overview of the following concepts.