2. Ray Theory of the Sound Field in the Ocean

Ray theory, in spite of its approximate nature, is a very effective method for the study of sound propagation at sufficiently high frequencies in inhomogeneous media such as the ocean. In this chapter we derive the basic equations of ray acoustics and give their solutions for a stratified ocean. In later chapters the ray approach is applied to waveguide sound propagation, reflection of sound from the sea surface, and some other problems.

2.1 Helmholtz Equation and Its Solution in Two Simple Cases

The propagation of monochromatic sound in the ocean is described by the Helmholtz equation [2.1] \((2.1.1)\)

\[ \Delta \psi + k^2 \psi = 0, \]  

where \(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\) is the Laplacian,

\[ \psi = \psi(x, y, z) \exp(-i \omega t) \] the acoustic velocity potential, \(k = \omega/c(x, y, z)\) the wave number, \(c(x, y, z)\) the sound velocity, \(\omega = 2\pi f = 2\pi/T\) the circular frequency, \(f\) the frequency, and \(T\) the sound-wave period. The particle velocity \(v\) and sound pressure \(p\) are expressed in terms of \(\psi\),

\[ v = \nabla \psi, \quad p = -\rho \frac{\partial \psi}{\partial t}. \]  

where \(\nabla\) is the gradient operator. We consider two of the simplest solutions of \((2.1.1)\) for the case of a homogeneous medium \((k\) is constant).

The first one is the spherical wave describing the field of an omnidirectional point source (a pulsating sphere of small radius)

\[ \psi = \frac{V_0 \exp(ikR)}{4\pi R}, \]  

\[1\] As has been shown in [2.1], at a frequency above 1 Hz the effect of the gravity and density gradients in the ocean may be neglected.
where \( R = (x^2 + y^2 + z^2)^{1/2} \). The sound source is assumed to be at the origin of coordinates, \( x = y = z = 0 \). In (2.1.3) and subsequently the factor \( \exp(-i\omega t) \) will be omitted. One can easily verify that (2.1.1) satisfies (2.1.3) by direct differentiation. In (2.1.3) \( V_0 = 4\pi a^2 v_0 \) is the volume velocity of the source (\( a \) is the radius of the oscillating sphere, \( v_0 \) the amplitude of its surface velocity).

Another simple and important solution of (2.1.1) is a plane wave

\[
\psi = A \exp[i(k_x x + k_y y + k_z z)],
\]

where \( A \) is the amplitude of the wave and \( k_x, k_y, \) and \( k_z \) are three arbitrary constants (components of the wave vector along the coordinate axes) satisfying the relation

\[
k_x^2 + k_y^2 + k_z^2 = k^2.
\]

The surfaces of constant phase (wave fronts) of the spherical wave (2.1.3) are spheres (\( R = \text{const} \)), and the rays which, by definition, are a family of lines orthogonal to the fronts are straight lines starting from the point \( R = 0 \). In the case of the plane wave (2.1.4) the fronts are planes, \( k_x x + k_y y + k_z z = \text{const} \), and the rays are the family of parallel straight lines orthogonal to these planes.

The solution of (2.1.1) in the form of a plane wave (2.1.4) is of great importance. In many cases, especially at sufficiently large distances from the source, the sound wave can be represented as a plane wave, or as a superposition of plane waves. This is obvious, for instance, for a spherical wave at large distances where the wave front curvature may be disregarded.

### 2.2 Refraction of Sound Rays

Consider first a horizontally stratified ocean where sound velocity only depends upon the depth \([c = c(z)]\), and the surface and bottom of the ocean are horizontal planes. Even with these simplifying assumptions one can only succeed in finding exact solutions of (2.1.1) in exceptional cases. Therefore, the ray acoustics approximation is widely used. The necessary (but not sufficient) condition for its application is that the product of the relative gradient of the sound velocity and the wave length must be small:

\[
\frac{\lambda}{c} \left| \frac{dc}{dz} \right| \ll 1.
\]  

2 It is reasonable to use the Laplacian expression in the spherical coordinates, taking into account that \( \psi \) only depends upon \( R \)

\[
\Delta \psi = \frac{\partial^2 \psi}{\partial R^2} + \frac{2}{R} \frac{\partial \psi}{\partial R}.
\]