SOME RECENT DEVELOPMENTS AND PROSPECTS IN FINITE DIFFERENCE METHODS.

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1. INTRODUCTION

By "finite differences" we may understand finite methods for modelling differential equations. In the classical theory of finite differences we seek (strong and generalised) consistency with differential equations, (weak and strong) convergence to solutions of differential equations and stability of the corresponding numerical processes (e.g. Richtmyer and Morton, 1967; Abbott, 1979; Rosinger, 1980) while striving to make optimal trade-offs between investment costs and running costs. This classical structure is currently undergoing an extensive "rethinking" process. The present contribution is intended only to outline some of the more immediately practical developments that have already arisen from this rethinking.

The main themes are the following:
1. The use of higher accuracy (in classical terms, third/fourth-order) methods,
2. The introduction of de-averaging (de-filtering) processes at the level of the differential equations,
3. The introduction of control components in system frames.

These developments are illustrated by examples.

2. THE USE OF METHODS OF HIGH-ORDER ACCURACY

The principle of generating high-order accuracy is simple and, if algorithmic constraints and their centring problems are put to one side, almost trivial. One works out the Taylor-series truncation error generated by any proposed difference scheme with a given centring (usually one that has some good accuracy/speed characteristics in the first place) and one then modifies the scheme so as to cancel-out the highest-order part of this truncation error. One then calculates the truncation error of the corrected scheme and modifies it again in order to cancel-out further errors, and so on. The realisation is complicated
by the wish to maintain those algorithmic structures that provided the original accuracy/speed advantages, so that the subtracting-out becomes correspondingly constrained, and by the need to carry through this programme in the presence of truncation errors that involve a very large number of terms. Thus the main two-dimensional production systems used in hydraulic engineering practice (e.g. those of RAND, DHI, EDF/SOGREAH) all make use of fast ADI algorithms which involve, in principle, between 100 and 200 correction terms, all of which must be cancelled out, one against the other, in order to attain third/fourth-order accuracy.

The systematisation of this process in hydraulics (coastal and offshore engineering, oceanology etc.) has occupied a considerable number of researchers over the last decade. A very readable description of one such systematisation and a review of earlier work is provided by Navon and Riphagen (1979). These authors also give references to studies of boundary conditions applicable to such high-order schemes, but they provide only a restricted stability analysis. The von Neumann stability of the linearised two-dimensional schemes and of the high-order one-dimensional schemes consistent with the Boussinesq equations has been discussed by Abbott, McCowan and Warren (1981). These last authors also provide an analysis of the numerical vorticity generated by such schemes while Leendertse et al (1981) provide an example whereby correction terms are introduced precisely in order to cancel-out the numerical vorticity (and thence estrophy) generation. The principles of this procedure date back, of course, to Arakawa (1966), who indeed treated the problem of conserving any integral invariant of the fluid motion. Since it is known, since Benney (1974), that the number of these invariants is infinite, there is an infinite scope for improvement in this direction. In the real world, sub species necessitatis, the choice of integral invariant is made on the basis of the requirements of the application.

A high-order accuracy is needed whenever the truncation errors of the scheme give rise to properties of their solutions that have a significant unrealistic physical interpretation. In order to make this statement less tautological, some examples are required. In the case of the propagation of dispersive wave trains, it was shown by Lighthill and Whitham (1954) that small perturbations in the energy balance of the train would cause considerable differences in the wave lengths and thence in the celerities of the individual waves. This as well appears from a truncation-error analysis of second-order mass and momentum difference equations, where the third order terms are seen to have the same form as and a comparable magnitude to the terms introduced by those vertical accelerations that close the equation system. The truncation error then provides an additional, pseudo-dispersive mechanism. Thus the influence of the third-order error is to change,