XI. Application to Stochastic Control

**Statement of the problem.** Suppose that the state of a system at time \( t \) is described by a stochastic integral \( X_t \) of the form

\[
(11.1) \quad dX_t = dX^u_t = b(t, X_t, u) dt + \sigma(t, X_t, u) dB_t,
\]

where \( X_t, b \in \mathbb{R}^n, \sigma \in \mathbb{R}^{n \times m} \) and \( B_t \) is \( m \)-dimensional Brownian motion. Here \( u \in \mathbb{R}^k \) is a parameter whose value we can choose at any instant in order to control the process \( X_t \). Thus \( u = u(t, \omega) \) is a stochastic process. Since our decision at time \( t \) must be based upon what has happened up to time \( t \), the function \( w \mapsto u(t, w) \) must (at least) be measurable wrt. \( \mathcal{F}_t \), i.e. the process \( u_t \) must be \( \mathcal{F}_t \)-adapted. Thus the right hand side of (11.1) is well-defined as a stochastic integral, under suitable assumptions on \( b \) and \( \sigma \). At the moment we will not specify the conditions on \( b \) and \( \sigma \) further, but simply assume that the process \( X_t \) satisfying (11.1) exists. See further comments on this in the end of this chapter.

Let \( \{X^t, x\}_{t \geq 0} \) be the solution of (11.1) such that \( X^t, x = x \), i.e.

\[
X^t, x = x + \int_0^t b(r, X^r, x, u) dr + \int_0^t \sigma(r, X^r, x, u) dB_r
\]

and let the probability law of \( X_s \) starting at \( x \) for \( s = t \) be denoted by \( Q^{t, x} \), so that

\[
Q^{t, x}[X^t, x \in E_1, \ldots, X^t, x \in E_K] = P^0[X^t, x \in E_1, \ldots, X^t, x \in E_K]
\]

for \( t < t_1, 1 \leq i \leq k \).

To obtain an easier notation we introduce

\[
Y_s = (t+s, X^t, x) \quad \text{for} \quad s > 0, \quad Y_0 = (t, x)
\]

and we observe that if we substitute in (11.1) we get the equation

\[
(11.3) \quad dY_t = dY^u_t = b(Y_t, u) dt + \sigma(Y_t, u) dB_t,
\]

i.e. the stochastic integral is time-homogeneous. (Strictly speaking, the \( u, b \) and \( \sigma \) in (11.3) are slightly different from the \( u, b \) and \( \sigma \) in (11.1).) The probability law of \( Y_s \) starting at \( y = (t, x) \) for \( s = 0 \) is (with abuse of notation) also denoted by \( Q^{t, x} = Q^y \).

We assume that a **cost function** (or performance criterion) has been given on the form
(11.4) \[ J(t,x,u) = E^t,x \left[ \int_t^\tau F(s,X_s,u)ds + K(\tau,X_\tau) \right] \]
or with \( y = (t,x) \),

\[ J^u(y) = E^y \left[ \int_0^\tau F^u(Y_s)ds + K(Y_\tau) \right], \quad \text{with} \quad F^u(y) = F(y,u), \quad J^u(y) = J(Y,u), \]

where \( K \) is a bounded "bequest" function, \( F \) is bounded and continuous and \( \tau \) is assumed to be the exit time of \( Y_s \) from some (fixed) open set \( G \subset \mathbb{R}^{n+1} \). Thus, in particular, \( \tau \) could be a fixed time \( t_0 \). We assume that \( E^y[\tau] < \infty \) for all \( y \in G \).

The problem is to find a control function \( u^* = u^*(t,\omega) \) such that

(11.5) \[ H(y) \overset{\text{def}}{=} \inf_{u(t,\omega)} \left\{ J^u(y) \right\} = J^u^*(y) \quad \text{for all} \quad y \in G \]

where the \( \inf \) is taken over all \( \mathcal{F}_t \)-adapted processes \( \{u_t\} \),

usually required to satisfy some extra conditions. Such a control \( u^* \) - if it exists - is called an \textit{optimal control}. Examples of types of control functions that may be considered:

1) Functions \( u(t,\omega) = u(t) \) i.e. not depending on \( \omega \). These controls are sometimes called deterministic or open loop controls.

2) Processes \( \{u_t\} \) which are \( \mathcal{M}_t \)-adapted, i.e. for each \( t \) the function \( \omega + u(t,\omega) \) is \( \mathcal{M}_t \)-measurable, where \( \mathcal{M}_t \) is the \( \sigma \)-algebra generated by \( \{X^u_s; s \leq t\} \). These controls are called closed loop or feedback controls.

3) The controller has only \textit{partial knowledge} of the state of the system. More precisely, to the controller's disposal are only (noisy) observations \( R_t \) of \( X_t \), given by a stochastic integral of the form

\[ dR_t = a(t,X_t)dt + \gamma(t,X_t)dB. \]

Hence the control process \( \{u_t\} \) must be adapted wrt. the \( \sigma \)-algebra \( \mathcal{N}_t \) generated by \( \{R_s; s \leq t\} \). In this situation the stochastic control problem is linked to the filtering problem (Chapter VI). In fact, if the equation (11.1) is linear and the cost function is integral quadratic (i.e. \( F \) and \( K \) are quadratic) then the stochastic control problem splits into a linear filtering problem and a corresponding deterministic control problem. This is called \textit{The Separation Principle}. See Example 11.4.