Chapter V. Modular Forms and the Minimal Compactification

0. Introduction

The major goal of this section is to lay the foundation for an arithmetic theory of Siegel modular forms and to construct an arithmetic minimal compactification of (the coarse moduli space of) $\mathbb{A}_g$, which mimics the Satake-Baily-Borel compactification of arithmetic quotients of bounded symmetric domains. These two are closely intertwined, as they must be.

We shall define modular forms as global sections of powers of $\omega$ over (any) $\mathbb{A}_g$. This definition is independent of the choice of the cone decomposition. Then we establish the $q$-expansion principle and the Koecher principle, to the effect that modular forms are determined by their $q$-expansions and that any meromorphic modular form regular on $\mathbb{A}_g$ is automatically holomorphic at the boundary, if the genus $g$ is $> 1$. The minimal compactification $\mathbb{A}_g^*$ is defined as the projective spectrum of the graded ring of modular forms. By its construction, $\mathbb{A}_g^*$ is a blowing-down of $\mathbb{A}_g$. Roughly speaking, in terms of the semi-abelian scheme $G$ over $\mathbb{A}_g$, the natural map $\mathbb{A}_g \to \mathbb{A}_g^*$ associates to a semi-abelian variety its abelian part. Some power of $\omega$ descends to an ample invertible sheaf on the minimal compactification. The necessary power comes from the non-trivial automorphisms of principally polarized abelian varieties of dimension $\leq g$. There are obvious analogues if we throw in principal level-$n$ structure. If we do so for $n \geq 3$ then $\omega$ itself descends to the minimal compactification. As a byproduct, it follows that the ring of integral Siegel modular forms is finitely generated over $\mathbb{Z}$.

As an application we prove a theorem that the $p$-adic monodromy of the "physical $p$-torsion points" of generic principally polarized abelian varieties in characteristic $p > 0$ is "as large as possible", namely $\text{GL}_g(\mathbb{Z}_p)$. Of course, this same method can be applied to show that the $p$-adic monodromy is big for modular varieties coming from abelian varieties, polarizations and endomorphism structures.

In chapter IV it was shown that for $n \geq 3$, $\mathbb{A}_{g,n}$ is in fact an algebraic space. The reader may ask whether it is indeed a scheme. Although we do not give a complete answer, we do supply a sufficient condition under which $\mathbb{A}_{g,n}$ is a
1. Modular Forms and Their $q$-Expansions

Let us fix a GL($X$)-admissible polyhedral cone decomposition $\{\sigma_\alpha\}$ of $C(X)$ as in chap. IV §2, and let $\bar{A}_g$ be the associated arithmetic toroidal compactification. Over $\bar{A}_g$ we have a canonical invertible sheaf $\omega = \det(T_G^*) = \det(\Omega_{\bar{G}/\bar{A}_g})$, corresponding to the highest order translation-invariant relative differentials of the semi-abelian variety $G$ over $\bar{A}_g$. Modular forms of weight $k$ are just global sections of $\omega^k$:

1.1. Definition. Suppose $M$ is an abelian group. A (Siegel) modular form of genus $g$ and weight $k$ with coefficients in $M$ regular at infinity is an element of $\Gamma(\bar{A}_g, \omega^k \otimes M)$.

1.2. Remarks. (a) In most applications $M$ will be a commutative ring, but for the proof of the $q$-expansion principle we need the full generality of the definition 1.1. As a functor in $M$, modular forms are left exact and commute with filtering inductive limits. (b) The readers surely have noticed that we have made a choice of a GL($X$)-admissible polyhedral cone decomposition $\{\sigma_\alpha\}$ of $C(X)$, so it seems that definition 1.1. depends on that choice, which is undesirable. In fact this definition is really independent of the choice of the cone-decomposition leading to $\bar{A}_g$, here is one way to see it: By taking refinements, we only have to show that for two GL($X$)-admissible polyhedral cone decompositions $\{\sigma_\alpha\}, \{\sigma'_\zeta\}$, one being a refinement of the other, we have $\Gamma(\bar{A}_g, \omega^k \otimes M) = \Gamma(\bar{A}_g, \omega^k \otimes M)$. The point is that the natural morphism map $\pi$ from $\bar{A}_g(\sigma_\alpha)$ to $\bar{A}_g(\sigma'_\zeta)$ induces a natural identification between the corresponding sets of global sections of $\omega^k$. This is a consequence of the facts that $\pi_*\mathcal{O} = \mathcal{O}$ and $R^i\pi_*\mathcal{O} = 0$ for all $i > 0$ universally. The first is immediate, and the second is because the locally in the étale topology $\pi$ can be explicitly described. In fact locally in the étale topology $\pi$ is isomorphic to the map between the torus embeddings attached to $\{\sigma_\alpha\}$ and $\{\sigma'_\zeta\}$: say $\{\sigma_\alpha\}$ is a refinement of $\{\sigma'_\zeta\}$, and $\pi_T : \text{Temb}\{\sigma_\alpha\} \to \text{Temb}\{\sigma'_\zeta\}$.