An upper bound is given for the maximum number, \( N_c \), of negative particles (fermions or bosons or a mixture of both) of charge \(-e\) that can be bound to an atomic nucleus of charge \(+ze\). If \( z \) is integral then \( N_c \leq 2z \). In particular, this is the first proof that \( \text{H}^- \) is not stable. For a molecule, \( N_c \leq 2Z + K - 1 \), where \( K \) is the number of atoms in the molecule and \( Z \) is the total nuclear charge.

PACS numbers: 03.65.Ge, 31.10.+z

The purpose of this note is to announce a theorem about \( N_c \), the full details of which will appear elsewhere.\(^9\) The theorem applies to any mixture of bound particles, with possibly different statistics, masses, and charges (as long as they are all negative), and even with possibly different magnetic fields acting on the various particles. (Naturally, symmetry requires that particles of the same type have the same mass, etc.) The theorem also applies to a molecule. The usual approximation that the nuclei be fixed (or infinitely massive) is important, but if they are not fixed a weaker theorem holds. The same theorems hold in the Hartree-Fock (restricted or unrestricted) and Hartree approximations to the ground-state energy.\(^7\)

Suppose that we have a molecule with \( K \) nuclei of charges \( z_1, \ldots, z_K > 0 \) (units are used in which the electron charge is unity) located at fixed, distinct positions \( \bar{R}_1, \ldots, \bar{R}_K \). The electric potential of these nuclei is

\[
V(\mathbf{x}) = \sum_{j=1}^{K} z_j |\mathbf{x} - \bar{R}_j|^{-1}.
\]

Let there be \( N \) negative particles with masses \( m_1, \ldots, m_N \) and charges \(-q_1, \ldots, -q_N < 0\) (in the usual case each \( q_i = 1 \)) and let each be subject to (possibly different) magnetic fields \( \vec{A}_1(\mathbf{x}), \ldots, \vec{A}_N(\mathbf{x}) \). (The generality of allowing nonintegral nuclear and negative particle charges may have some physical relevance because, as pointed out to me by W. Thirring, particles in solids such as semiconductors may have nonintegral effective charges due to dielectric effects.) The Hamiltonian is

\[
H_N = \sum_{j=1}^{N} \left( T_j - q_j V(\mathbf{x}_j) \right) + \sum_{i<j} q_i q_j |\mathbf{x}_i - \mathbf{x}_j|^{-1}.
\]

Here, \( T_j \) is the kinetic energy operator for the \( j \)th particle and it is one of the following (possibly different for different \( j \)) two types (nonrelativistic or relativistic):

\[
T_j = |\vec{p}_j - q_j \vec{A}_j(\mathbf{x})/c|^2/2m_j, \quad T_j = (|\vec{p}_j - q_j \vec{A}_j(\mathbf{x})|^2 + m_j^2 c^4)^{1/2} - m_j c^2.
\]
Let \( q \) denote the maximum of the \( q_j \), let \( Q = \sum_j q_j \) be the total negative charge, and let \( z = \sum_j q_j \) be the total nuclear charge. Let \( E_n \) denote the ground-state energy of \( H_q \) [\( \pm \text{inf} \text{spec}(H_q) \)].

**Theorem 1.**—If the above system is bound (meaning that \( E_n \) is an eigenvalue of \( H_q \)) then, necessarily,

\[
Q < 2Z + qK.
\] (5)

In the atomic case (\( K = 1 \)) this can be strengthened to

\[
Q < 2Z + \sum_j q_j^2 / Q.
\] (6)

The strict inequality in Eqs. (5) and (6) is important; in the atomic case with fixed nuclear charge, let \( E_n \) denote the ground-state energy of \( H_q \) with \( z \) fixed, and let \( K = 1 \).

\[
E_n < Z + qK + \epsilon.
\] (7)

For a hydrogen atom (\( Z = 0 \)), Eq. (7) implies that \( N > 2 \). In the case where the two electrons can, in fact, be bound. \( H^- \) is not stable. This result had not been proved before, although there exist partial results in this direction.

Theorem 2.—If the above system is bound and if, in addition, \( E_n < E_{n,j} \) for all \( j = 1, \ldots, N \), then Eq. (5) [respectively Eq. (6) for \( K = 1 \)] holds.

The proof of the theorems is simple enough to be given in an elementary quantum mechanics course—at least in the atomic case with fixed nuclear charge and with \( T_j = p_j^2 / 2m_j \) (all \( j \)). The proof (ignoring some technical fine points) in this atomic case is the following: Take \( h^2 / 2m = 1 \) and \( R = 0 \). Pick some \( j \) and write \( H_n = H_{n,j} + \hbar \), where \( H_{n,j} \) is the Hamiltonian for the remaining \( N - 1 \) particles and

\[
\hbar_j = T_j - q_j z |\vec{x}_j|^{-1} + \sum_{k \neq j} |\vec{x}_j - \vec{x}_k|^{-1} q_j q_k.
\] (8)

Assume that the system is bound and let \( \psi \) be the ground state (which is real). Multiply the Schrödinger equation, \( H_n \psi = E_n \psi \), by \( |\vec{x}_j| \psi \) and integrate over all \( N \) variables. Let \( X_j \) denote the set of \( N - 1 \) variables other than \( \vec{x}_j \). For the \( H_{n,j} \) term, do the \( d^{N-1}X_j \) integration first; by the variational principle, the \( X_j \) integral is, for each fixed \( \vec{x}_j \), not less than \( E_{n,j} \) times the same integral without \( H_{n,j} \). This inequality is preserved after the \( \vec{x}_j \) integration since \( |\vec{x}_j| \) is a positive weight. Thus

\[
\langle |\vec{x}_j| \psi | H_{n,j} \psi \rangle \geq E_{n,j} \langle |\vec{x}_j| \psi | \psi \rangle.
\] (9)

Recalling the easily proved fact that \( E_n \leq E_{n,j} \), we have

\[
\langle |\vec{x}_j| \psi | \hbar_j \psi \rangle < 0.
\] (10)

The claim is that Eq. (10) cannot hold for all \( j \) if condition (6) is violated.

First, the term \( t_j = \langle |\vec{x}_j| \psi | \hbar_j \psi \rangle \) is positive. To see this, do the \( \vec{x}_j \) integration first and note that it then suffices to prove the following for any function, \( f \), of one variable:

\[
t = \int |\vec{x} f (\vec{x}) v^2 f (\vec{x}) d^3 x > 0.
\] (11)

Write \( g (\vec{x}) = |\vec{x} f (\vec{x}) \) and integrate by parts:

\[
t = \int \{ v g (\vec{x}) \} \{ |\vec{x}|^{-2} v g (\vec{x}) + k (\vec{x}) v |\vec{x}|^{-1} \} d^3 x
\]

\[
= \int \{ v g (\vec{x}) \}^2 |\vec{x}|^{-1} - \frac{1}{2} g (\vec{x}) v^2 |\vec{x}|^{-1} \} d^3 x > 0
\]

since \( v^2 |\vec{x}|^{-1} \leq 0 \). The fact that \( g \) \( v g \) \( v g \) \( v g \) together with another integration by parts, was used for the second term.)

The second term in \( t \) is easy:

\[
A_j = q_j \langle |\vec{x}_j| \psi | V \vec{x}_j \rangle |\psi \rangle = q_j \zeta (\psi | \psi \rangle = q_j \zeta,
\]

with the assumption that \( \psi \) is normalized.

The third term is

\[
R_j = \int \vec{x} |\vec{x}_j| \sum_{k \neq j} q_k q_j |\vec{x}_j - \vec{x}_k|^{-1} d^3 x,
\]

where \( X \) denotes all the \( N \) variables.

If there is binding then Eq. (10) holds for all \( j \) and hence, summing over \( j \) and using \( t_j > 0 \), we have that \( A = \sum_j A_j > \sum_j R_j = R \). On the one hand,