Chapter 7. Smirnov and Lavrentiev Domains

7.1 An Overview

We now consider domains with rectifiable boundaries in more detail. Let $f$ map $\mathbb{D}$ conformally onto the inner domain $G$ of the Jordan curve $J$. We have shown in Section 6.3 that $J$ is rectifiable if and only if $f'$ belongs to the Hardy space $H^1$.

We call $G$ a Smirnov domain if

$$\log |f'(z)| = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|\zeta - z|^2} \log |f'(\zeta)| \, d\zeta \quad \text{for} \quad z \in \mathbb{D}. \quad (1)$$

In the language of Hardy spaces this means that $f'$ is an "outer function". This is not always true, and the Smirnov condition has not been completely understood from the geometrical point of view. See e.g. Dur70, p. 173 for its significance for approximation theory.

Zinsmeister has shown (Theorem 7.6) that $G$ is a Smirnov domain if $J$ is Ahlfors-regular, i.e. if

$$A(\{z \in J : |z - w| < r\}) \leq M_1 r \quad \text{for} \quad w \in \mathbb{C}, \quad 0 < r < \infty \quad (2)$$

for some constant $M_1$. The condition of Ahlfors-regularity appears in many different contexts, for instance in G. David's work on the Cauchy integral operator.

We call $G$ a Lavrentiev domain if

$$A(J(a, b)) \leq M_2 |a - b| \quad \text{for} \quad a, b \in J, \quad (3)$$

where $J(a, b)$ is the shorter arc of $J$ between $a$ and $b$. This is the quasicircle condition (5.4.1) with diameter replaced by length, and Lavrentiev domains are the Ahlfors-regular quasidisks (Proposition 7.7). Tukia has proved (Theorem 7.9) that Lavrentiev curves can be characterized as the bilipschitz images of circles.

A different characterization was given by Jerison and Kenig (Theorem 7.11): A domain is a Lavrentiev domain if and only if it is a Smirnov quasidisk such that $|f'|$ satisfies the Coifman-Fefferman ($A_\infty$) condition. This is closely connected to the Muckenhoupt ($A_p$) and ($B_q$) conditions, also to the space BMOA of analytic functions of bounded mean oscillation, the dual space of $H^1$ by a famous result of Fefferman. We shall only quote some results of this
rich theory (see e.g. Gar81) which has close connections to stochastic processes (see e.g. Durr84).

Let again \( I(re^{it}) = \{ e^{i\theta} : |\theta - t| \leq \pi(1 - r)\} \). The condition
\[
(4) \quad \frac{1}{A(I(z))} \int_{I(z)} |f'(\zeta)|^q d\zeta \leq M_3|f'(z)|^q \quad (z \in \mathbb{D})
\]
is related to the \((B_q)\) condition and will appear repeatedly in this chapter:

a) If \((4)\) holds for some \( q > 0 \) then \( G \) is a Smirnov domain (Proposition 7.5) and even \( \log f' \in BMOA \) (Proposition 7.13).

b) If \( J \) is Ahlfors-regular then \((4)\) is satisfied with \( q < 1/3 \) (Theorem 7.6).

c) The quasidisk \( G \) is a Lavrentiev domain if and only if \((4)\) is true for \( q = 1 \) (Theorem 7.8). Then \((4)\) automatically holds for some \( q > 1 \) (Corollary 7.12) by a result of Gehring.

### 7.2 Integrals of Bloch Functions

It will be convenient first to consider integrals
\[
(1) \quad h(z) = \int_0^z (g(\zeta) - g(0))\zeta^{-1} d\zeta \quad (z \in \mathbb{D})
\]
of Bloch functions. These will also be needed in Section 10.4. Let \( K_1, K_2, \ldots \) denote suitable absolute constants and \( M_1, M_2, \ldots \) other positive constants.

**Proposition 7.1.** If \( g \in \mathcal{B} \) then \( h \) is continuous in \( \overline{\mathbb{D}} \) and, for \( |z| \leq 1, \ 0 < |t| \leq \pi \),
\[
(2) \quad \left| \frac{h(e^{it}z) - h(z)}{it} - g \left( \left( 1 - \frac{|t|}{\pi} \right) z \right) + g(0) \right| \leq K_1\|g\|_{\mathcal{B}}.
\]

**Proof.** It follows from (4.2.4) that \( h'(z) = O\left((1 - |z|)^{-1/2}\right) \) as \( |z| \to 1 \). Hence \( h \) is (Hölder-) continuous in \( \overline{\mathbb{D}} \); see Dur70, p. 74. In order to prove (2) we may assume that \( g(0) = 0, 1/2 < |z| < 1 \) and \( 0 < |t| \leq \pi/2 \); see (4.2.4). Now let \( 0 < t \leq \pi/2 \) and \( |\tau| \leq t \). We choose \( \sigma \) such that \( t = \pi(1 - e^{-\sigma}) \). Integration by parts shows that
\[
\int_{e^{-\sigma}z}^{e^{i\tau}z} g'(\zeta) \log \frac{e^{i\tau}z}{\zeta} d\zeta = h(e^{i\tau}z) - h(e^{-\sigma}z)(\sigma + i\tau)g(e^{-\sigma}z).
\]
It follows from (4.2.1) and from
\[
\left| \log \frac{e^{i\tau}z}{\zeta} \right| \leq \log \frac{1}{|\zeta|} + \arg \frac{e^{i\tau}z}{\zeta} \leq K_2(1 - |\zeta|) \quad \text{for} \quad \zeta \in [e^{-\sigma}z, e^{i\tau}z]
\]
that the integrand is bounded by \( K_2\|g\|_{\mathcal{B}} \). Hence