21 Covariance Function Matrices

It is actually difficult to characterize directly a covariance function matrix. This becomes easy in the spectral domain on the basis of Cramer's generalization of the Bochner theorem, which is presented in this chapter. We consider complex covariance functions.

Covariance function matrix

The matrix \( C(h) \) of direct and cross covariance functions of a vector of complex random functions (with zero means without loss of generality)

\[
z(x) = \begin{pmatrix} Z_1, \ldots, Z_i, \ldots, Z_N \end{pmatrix}^\top \quad \text{with} \quad \mathbb{E}[z(x)] = 0
\]

is defined as

\[
C(h) = \mathbb{E} \left[ z(x) z(x+h)^\top \right]
\]

The covariance function matrix is a Hermitian positive semi-definite function, that is to say, for any set of points we have \( x^n \in \mathcal{D} \) and any set of complex weights \( w_i^n \)

\[
\sum_{i=1}^N \sum_{j=1}^N \sum_{\alpha=0}^n \sum_{\beta=0}^n w_i^\alpha \overline{w}_j^\beta C_{ij}(x_\alpha - x_\beta) \geq 0
\]

A Hermitian matrix is the generalization to complex numbers of a real symmetric matrix. The diagonal elements of a Hermitian \( N \times N \) matrix \( A \) are real and the off-diagonal elements are equal to the complex conjugates of the corresponding elements with transposed indices: \( a_{ij} = \overline{a}_{ji} \).

For the matrix of direct and cross covariance functions of a set of complex variables this means that the direct covariances are real, while the cross covariance functions are generally complex.

Cramer's theorem

Following a generalization of Bochner's theorem due to Cramer [37] (see also [68], [208]), each element \( C_{ij}(h) \) of a matrix \( C(h) \) of continuous direct and cross covariance functions has the spectral representation

\[
C_{ij}(h) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{i\omega^\top h} dF_{ij}(\omega)
\]

H. Wackernagel, *Multivariate Geostatistics* © Springer-Verlag Berlin Heidelberg 1995
where the $F_{ij}(\omega)$ are spectral distribution functions and $\omega$ is a vector of the same dimension as $h$. The diagonal terms $F_{ii}(\omega)$ are real, non decreasing and bounded. The off-diagonal terms $F_{ij}(\omega)$ ($i \neq j$) are in general complex valued and of finite variation. Conversely, any matrix of continuous functions $C(h)$ is a matrix of covariance functions, if the matrices of increments $\Delta F_{ij}(\omega)$ are Hermitian positive semi-definite for any block $\Delta \omega$ (using the terminology of [173]).

**Spectral densities**

If attention is restricted to absolutely integrable covariance functions, we have the following representation

$$C_{ij}(h) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{i\omega^T h} f_{ij}(\omega) \, d\omega$$

This Fourier transform can be inverted and the *spectral density functions* $f_{ij}(\omega)$ can be computed from the cross covariance functions

$$f_{ij}(\omega) = \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-i\omega^T h} C_{ij}(h) \, dh$$

The matrix of spectral densities of a set of covariance functions is positive semi-definite for any value of $\omega$. For any pair of random functions this implies the inequality

$$|f_{ij}(\omega)|^2 \leq f_{ii}(\omega) f_{jj}(\omega)$$

With functions that have spectral densities it is thus simple to check whether a given matrix of functions can be considered as a covariance function matrix.

**EXERCISE 21.1** Compute the spectral density of the exponential covariance function (in one spatial dimension) $C(h) = b e^{-a|h|}$, $b > 0$, $a > 0$ using the formula

$$f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega h} C(h) \, dh$$

**EXERCISE 21.2** Show (in one dimensional space) that the function

$$C_{ij}(h) = e^{-\left(\frac{a_i + a_j}{2}\right)|h|}$$

can only be the cross covariance function of two random functions with an exponential cross covariance function $C_i(h) = e^{-a_i|h|}$, $a_i > 0$, if $a_i = a_j$. 