7. Logics with Fixed-Point Operators

In Chapter 6 we have introduced logics with fixed-point operators in the context of computational complexity and shown that they capture important complexity classes. In the present chapter we study the (finite) model theory of fixed-point logics. Though we sometimes refer to Chapter 6, we repeat the relevant definitions and results to provide an independent approach.

Throughout this chapter all structures will be finite. Equivalence of formulas means equivalence with respect to all finite structures.

7.1 Inflationary and Least Fixed-Points

We prove some basic facts about fixed-points and, besides inflationary fixed-point logic, present an important sublogic, namely least fixed-point logic. We state two major results, the proofs being postponed to the next section: Inflationary and least fixed-point logic have the same expressive power, and every formula is equivalent to a formula containing at most one fixed-point application — a fact that extends Corollary 6.5.3 from ordered structures to arbitrary ones.

We observed in Chapter 1 that many important notions and global relations cannot be expressed by first-order formulas. An example is given by the reflexive and transitive closure of the edge relation in a graph (cf. 1.3.7). In the vocabulary \( \tau := \{E\} \) for graphs consider the formula

\[
\chi(x, y, X) := (x = y \lor \exists z (Xxz \land Ezy)).
\]

It gives rise to a sequence of sets defined by

\[
X_0 := \emptyset, \quad X_{n+1} := \{(x, y) \mid \chi(x, y, X_n)\}.
\]

In other terms: Let \( G = (G, E^G) \) be a graph. Look at the function \( F^x \) on the power set of \( G \times G \), \( F^x : \text{Pow}(G^2) \to \text{Pow}(G^2) \), given by

\[
F^x(U) := \{(a, b) \in G^2 \mid G \models \chi[a, b, U]\}
\]

for \( U \subseteq G \times G \). Then (*) corresponds to the sequence of sets
Note that $F; = \{(a,b) \in G^2 \mid d(a, b) < n\}$, where $d(a, b)$ denotes the distance between $a$ and $b$ in $G$ (see 0.1). For $F^\infty := \bigcup_{n \geq 0} F_n^\infty$ we have $F(F^\infty) = F^\infty$, that is, $F^\infty$ is a fixed-point of $F^\infty$ and

$$F^\infty = \{(a, b) \in G^2 \mid d(a, b) < \infty\}.$$ 

In particular, for $a \neq b$,

$$a, b \text{ are connected by a path in } G \text{ iff } (a, b) \in F^\infty,$$

and

$$G \text{ is connected iff } F^\infty = G \times G.$$ 

The relations $F_n^\infty$ that lead to the fixed-point $F^\infty$ are first-order definable, but in general, $F_n^\infty$ is not. In the fixed-point logics we are going to introduce the relation $F^\infty$ will be definable, too; in particular, the connectivity of graphs will be expressible.

Let us first study some aspects of the process above on a more abstract level. Fix a finite set $M$. A function $F : \text{Pow}(M) \to \text{Pow}(M)$ gives rise to a sequence of sets

$$\emptyset, F(\emptyset), F(F(\emptyset)), \ldots$$

Denote its members by $F_0, F_1, \ldots$, i.e. $F_0 = \emptyset$ and $F_{n+1} = F(F_n)$. $F_n$ is called the $n$-th stage of $F$. Suppose that there is an $n_0 \in \mathbb{N}$ such that $F_{n_0+1} = F_{n_0}$, that is, $F(F_{n_0}) = F_{n_0}$. Then $F_m = F_{n_0}$ for all $m \geq n_0$. We denote $F_{n_0}$ by $F_{\infty}$ and say that the fixed-point $F_{\infty}$ of $F$ exists. In case the fixed-point $F_{\infty}$ does not exist, we agree to set $F_{\infty} := \emptyset$.

$F$ is inductive if $F_0 \subseteq F_1 \subseteq \ldots$

**Lemma 7.1.1.** (a) *If $F_{\infty}$ exists then $F_{\infty} = F_{\|M\| - 1}$.*

(b) *If $F : \text{Pow}(M) \to \text{Pow}(M)$ is inductive then $F_{\infty}$ exists and $F_{\infty} = F_{\|M\|}$.***

**Proof.** Part (a) coincides with (a) of 6.1.1.

(b) By assumption, $F_0 \subseteq F_1 \subseteq \ldots \subseteq M$. Since $M$ has $\|M\|$ elements, this sequence must get constant not later than with $F_{\|M\|}$. \qed

$F$ is inflationary if for all $X \subseteq M$

$$X \subseteq F(X),$$

and monotone if for all $X, Y \subseteq M$

$$X \subseteq Y \subseteq M \text{ implies } F(X) \subseteq F(Y).$$