13. Klein's Paradox

In the following, we want to concern ourselves with the scattering of an electron with energy $E$ and momentum $p = p_z$ at an infinitely extended potential step (Fig. 13.1). First we shall study this problem from the point of view of the one-particle interpretation of the Dirac equation and then, in Example 13.1, we shall look at the same problem using the framework of hole theory, understanding better the resulting situation, which looks paradoxical at first sight.\footnote{O. Klein: Z. Phys. 53, 157 (1929).}

For the free electron we have $(E/c)^2 = p^2 + m_0c^2$, whereas in the presence of the constant potential,\footnote{W. Greiner, Relativistic Quantum Mechanics © Springer-Verlag Berlin Heidelberg 1997}

$$\left(\frac{E - V_0}{c}\right)^2 = \bar{p}^2 + m_0c^2 \tag{13.1}$$

is valid, where $\bar{p}$ denotes the momentum of the electron inside the potential. The Dirac equation and its adjoint then read

$$\left\{ \frac{E - eV}{c} - \beta m_0c \right\} \psi + i\hbar \sum_{k=1}^{3} \hat{\alpha}_k \frac{\partial \psi}{\partial x_k} = 0 \ , \tag{13.2a}$$

$$\bar{\psi} \left\{ \frac{E - eV}{c} - \beta m_0c \right\} + i\hbar \sum_{k=1}^{3} \frac{\partial \bar{\psi}}{\partial x_k} \hat{\alpha}_k = 0 \ . \tag{13.2b}$$

We now assume that

$$eV = V_0 \quad \text{for} \quad z > 0 \ ,$$

$$eV = 0 \quad \text{for} \quad z < 0 \ ,$$

and that the incoming wave is given by

$$\psi_i = u_i \exp \left\{ \frac{i}{\hbar} (pz - Et) \right\} \ , \tag{13.3}$$

so that, inserting (13.3) into (13.2a) and using $\hat{\alpha} = \hat{\alpha}_3$ it follows that

$$\left\{ \frac{E}{c} - \hat{\alpha} p - \beta m_0c \right\} u_i = 0 \ . \tag{13.4}$$
Since we require \( u_1 \neq 0 \), then because of \( \hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\alpha} = 0 \) we conclude that

\[
\frac{E^2}{c^2} = p^2 + m_0^2c^2 ,
\]  

(13.5)

and, moreover, due to our interest in the incoming electrons we choose \( E > 0 \). The momentum of the reflected wave must be \(-p\), whereas the momentum \( \overline{p}\) of the transmitted wave is given by (13.1). For small \( V_0, \overline{p}\) is positive, so that in the first instance we can set

\[
\psi_t = u_t \exp \left\{ \frac{i}{\hbar}(-pz - Et) \right\} , \quad \psi_t = u_t \exp \left\{ \frac{i}{\hbar}(\overline{p}z - Et) \right\}
\]  

(13.6)

and therefore, due to (13.2a),

\[
\left\{ \frac{E}{c} + \hat{\alpha}p - \hat{\beta}m_0c \right\} u_t = 0 \quad \text{and} \quad \left\{ \frac{E - V_0}{c} - \hat{\alpha}\overline{p} - \hat{\beta}m_0c \right\} u_t = 0 .
\]  

(13.7)

The total wave function must be continuous at the boundary, i.e. for \( z = 0 \)

\[
u_i + u_t = u_t
\]  

(13.8)

must be valid. From (13.4) and (13.8) therefore follows

\[
\left( \frac{E}{c} - \hat{\beta}m_0c \right)(u_i + u_t) = +\hat{\alpha}p(u_i - u_t) ,
\]  

(13.9)

and with (13.7) and (13.8) we get

\[
\left( \frac{E}{c} - \hat{\beta}m_0c \right)(u_i + u_t) = \left( \frac{V_0}{c} + \hat{\alpha}\overline{p} \right)(u_i + u_t) .
\]  

(13.10)

Thus we have

\[
\left\{ \frac{V_0}{c} + \hat{\alpha}\overline{p} \right\}(u_i + u_t) = +\hat{\alpha}p(u_i - u_t)
\]  

(13.11)

or

\[
\left\{ \frac{V_0}{c} + \hat{\alpha}(p + \overline{p}) \right\} u_t = -\left\{ \frac{V_0}{c} - \hat{\alpha}(p - \overline{p}) \right\} u_t .
\]  

(13.12)

We multiply both sides by \( V_0/c - \hat{\alpha}(p + \overline{p}) \), which, because of \( \hat{\alpha}^2 = 1 \) and with (13.1) and (13.5), leads to

\[
u_t = \frac{(2V_0/c)(-E/c + \hat{\alpha}p)}{V_0^2/c^2 - (p + \overline{p})^2}u_t \equiv ru_i .
\]  

(13.13)

Analogously we find for the adjoint amplitude

\[
u_t^\dagger = ru_i^\dagger ,
\]  

(13.14)

i.e.

\[
u_t^\dagger u_t = \left( \frac{2V_0/c}{V_0^2/c^2 - (p + \overline{p})^2} \right)^2 u_t^\dagger \left( \frac{-E}{c} + \hat{\alpha}p \right)^2 u_i ,
\]  

(13.15)

so that using the identity