In numerous technical applications, as well as some theoretical considerations, an enticing problem is encountered. In its most practical (and real) form the problem may be that of designing a radar (or sonar) transmitter waveform with a given spectrum such that its peak factor is a minimum. Peak factor is defined here as the range of the waveform values divided by their root-mean-square. This is important in radar and sonar, because one often wants to radiate a maximum amount of signal power, of a prescribed spectral shape, with a given peak power limitation on the transmitter.

In many applications, the prescribed spectral shape is a flat (or “white”) spectrum. One waveform that has such a flat spectrum is a sharp impulse; but in this case all the power is concentrated at one point in time, leading to a maximum peak power. What we want is a minimum peak power. The problem also occurs in talking computers and electronic speech synthesizers in general: synthetic speech made from sharp impulses sounds harsh and reedy; it suffers from a none-too-pleasant “electronic accent”. By contrast, computer speech from low peak-factor waveforms sounds smoother.

How can we avoid sharp pulses without modifying the power spectrum? Obviously, we still have the phase angles of the Fourier coefficients to play with.

For a given amplitude spectrum (magnitudes of Fourier transform coefficients), how does one choose the phase angles of the Fourier coefficients in order to achieve the smallest range of magnitudes in the corresponding inverse Fourier transform?

The results on waveforms described in this chapter are also applicable to antenna directivity problems. Specifically, a low peak factor corresponds to an antenna with a wide radiation or receiving directivity pattern.

Another antenna problem solved by number theory is one of “sparse arrays”, namely minimum-redundancy antennas. Such antennas play an impor-

\[\text{Range} = \frac{\text{peak power}}{\text{root-mean-square}}\]

\[\text{Peak factor} = \frac{\text{range}}{\text{root-mean-square}}\]

\[\text{Flat spectrum} = \text{white spectrum}\]

\[\text{Sharp impulse} = \text{maximum peak power}\]

\[\text{Synthetic speech} = \text{electronic accent}\]

\[\text{Low peak-factor waveforms} = \text{smooth}\]

\[\text{Sparse arrays} = \text{minimum-redundancy antennas}\]

\[\text{Number theory} = \text{antenna directivity problems}\]

\[\text{Minimum-redundancy antennas} = \text{sparse arrays}\]

\[\text{Wolfgang Pauli}\]

\[\text{Dirac functions}\]

\[\text{Quantum electrodynamics (QED)}\]

\[\text{Spin and antimatter}\]

\[\text{Lorentz invariance}\]

\[\text{Elementary particle}\]

\[\text{Wolfgang Pauli}\]
tant role in astrophysics and ocean surveillance, where the individual antenna elements (e.g., steerable parabolic "dishes" or submerged hydrophones) are very expensive or costly to control. In such cases, one wishes to construct arrays with the smallest number of elements for a given unambiguous target resolution (Sect. 28.5). Minimum redundancy arrays have also become important in real-time diagnostic tomography.

28.1 Special Phases

Early waveforms with low peak factors were found by the author and V. A. Vyssotsky by Monte Carlo computations: several thousand periodic waveforms of a given power spectrum, but different sets of random phase angles, were generated on the computer and sorted according to increasing peak factor. The best waveforms thus found had a peak factor several times smaller than the zero-phase impulse.

Later, the author [28.1] developed a formula for the phase angles $\alpha_n$ of low peak-factor waveforms with a given power spectrum $P_n$, based on asymptotic spectra of certain frequency-modulated signals:

$$\alpha_n = \alpha_1 - \frac{2\pi}{P} \sum_{k=1}^{n-1} (n-k)P_k$$

where $P$ is the total power:

$$P := \sum_{k=1}^{N} P_k$$

For flat power spectra, $P_k = \text{const}$, (28.1) can be simplified to

$$\alpha_n = \alpha_1 - \frac{\pi}{N} n^2$$

a quadratic dependence of phase on harmonic number (frequency) $n$.

Figures 28.1 and 28.2 illustrate the reduction in peak factor achieved with formula (28.1) for a nonflat power spectrum.

If the low peak-factor waveform is desired to be symmetric in time, then phase angles are restricted to 0 or $\pi$. Formula (28.1) is then replaced by [28.1]:

$$\alpha_n = \pi \left[ \sum_{k+1}^{n-1} (n-k) \frac{P_k}{P} \right]$$

or, for flat spectra,

$$\alpha_n = \pi \left[ \frac{n^2}{2N} + c \right]$$

where $c$ is a constant that can be adjusted to minimize rounding effects due to the floor function (Gauss bracket) employed in (28.5).