THE $N^{5/3}$ LAW FOR BOSONS

Elliott H. Lieb 1

Departments of Mathematics and Physics, Princeton University, Princeton, NJ 08540, USA

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Non-relativistic negative bosons interacting with infinite mass positive particles via Coulomb forces are shown to be unstable in the sense that $E_0 < -CN^{5/3}$. This agrees with the previously known lower bound $E_0 \geq -AN^{5/3}$.

In a celebrated series of papers [1–3], Dyson and Lenard proved that matter is quantum-mechanically stable under the action of Coulomb forces provided all the particles of at least one sign of charge (say negative) are fermions. In other words, the ground state energy $E_0$ satisfies

$$E_0 \geq -A_{f}N \quad \text{(negative fermions)},$$

where $N$ is the number of negative particles. It is not necessary to assume neutrality or that the positive particles have finite mass. It is necessary to assume that all the charges are bounded however. (1) was subsequently rederived by Federbush [4] and by Lieb and Thirring [5]. The best current value [6] is

$$A_{f} \leq -22.24 \text{ Ry},$$

for electrons and protons, and with the assumption of neutrality.

If all the particles are bosons or, what is the same thing, are not subject to any statistics, the best available lower bound [1–7] is

$$E_0 \geq -A_{b}N^{5/3} \quad \text{(all bosons)},$$

with [6]

$$A_{b} \leq 14.01 \text{ Ry},$$

when the positive and negative particles have charges $\pm e$ and the negative particle mass is $m_e$. The positive particles can have infinite mass.

Dyson [8] then proved, by a complicated variational calculation, that in the boson case

$$E_0 \leq -BN^{7/5},$$

if all particles have finite mass. It was conjectured [1,8] that $N^{7/5}$ is the correct law for bosons and not $N^{5/3}$. While this question might have only moderate practical importance (it would be relevant for $\pi$-mesons and $^4$He nuclei, for example), it has great theoretical importance and it is to be hoped that its solution will soon be forthcoming. It is interesting because at this point there is no simple, compelling physical argument, as distinguished from a computational argument [9], why the $N^{7/5}$ law is correct. Subtle correlation effects are yet to be understood fully and rigorously.

The purpose of this paper is to add a minor commentary on the problem. By means of a simple variational calculation, it will be shown that the $N^{7/5}$ law is indeed correct if the positive particles have infinite mass and charge $z|e| > 0$, i.e.

$$E_0 \leq -CN^{5/3},$$

If the negative particles have mass $m_e$ and charge $-|e|$, and if the system is neutral, then

$$C \geq (1/108)z^{4/3} \text{ Ry}.$$

Thus, the limits $N \to \infty$ and the mass of the positive particles $\to \infty$ are not interchangeable if the $N^{7/5}$ conjecture is correct.

We note in passing that (1) alone does not imply that $e_0 = \lim_{N \to \infty} E_0/N$ exists. However, the method

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developed to prove the existence of the thermodynamic limit \[10\] also proves that this limit exists. Likewise (3) and (6a) do not imply that 
\[
\lim_{N \to \infty} E_0 / N^{3/3}
\]
exists. The aforementioned method \[10\] is not suitable for this task and we do not know an adequate substitute.

For simplicity of exposition we assume there is one kind each of positive and negative particle. The \(N\) negative particles have a mass \(1/2\) and charge \(-1\). If \(h^2 = 1\), then one Rydberg \(= 1/4\). The \(K\) infinite mass positive particles have charge \(z > 0\) and we assume \(N = Kz\) (neutrality) with \(K = 8n^3\), \(n\) an integer. Other cases can easily be handled by this method but we omit details for simplicity. The hamiltonian for the negative particles is

\[
H_{N,R} = -\sum_{i=1}^{N} \left( \Delta_i + V_R(r_i) \right) + \sum_{1 \leq i < j \leq N} |r_i - r_j|^{-1} + U(R),
\]

where \(R = \{R_1, \ldots, R_K\}\) is the collection of fixed coordinates of the positive particles and

\[
V_R(r) = z \sum_{j=1}^{K} |r - R_j|^{-1},
\]

\[
U(R) = z^2 \sum_{1 \leq i < j < k} |R_i - R_j|^{-1}.
\]

We want to find a normalized \(\psi(r_1, \ldots, r_N)\) and \(R\) (depending on \(N\)) such that \(\langle \psi | H_{N,R} | \psi \rangle \leq -CN^{3/3}\).

\(\psi\) is chosen to be a simple product of identical functions:

\[
\psi(r_1, \ldots, r_N) = \prod_{i=1}^{N} \phi_\lambda(r_i).
\]

\(\phi_\lambda\) depends on the parameter \(\lambda\) (which will turn out to be proportional to \(N^{1/3}\)) as follows:

\[
\phi_\lambda(r) = \lambda^{3/2} g(\lambda r),
\]

where \(g(r)\) is the fixed, normalized function

\[
g(r) = f(x) f(y) f(z), \quad r = (x, y, z),
\]

and

\[
f(x) = \sqrt{3/2} \left[ 1 - |x| \right], \quad |x| \leq 1,
\]

\[
= 0, \quad |x| > 1.
\]

The explicit choice in (12) is neither important nor optimal. With

\[
T = \int |\nabla g(r)|^2 \, d^3r = 9,
\]

\[
\langle \psi | H_{N,R} | \psi \rangle \leq \lambda^2 N T + \lambda W(N, R),
\]

\[
W(N, R) = \frac{1}{2} N^2 \int \int g^2(r) g^2(r') |r - r'|^{-1} \, d^3r \, d^3r'
\]

\[
- N \int g^2(r) V_R(r) \, d^3r + U(R).
\]

There is \(\leq\) in (14) because we should have \(N(N - 1)\) instead of \(N^2\) in (15). We claim \(R\) can be chosen so that

\[
W(N, R) \leq -(12)^{-1/2} z^{2/3} N^{4/3}.
\]

If so, (6) is proved by minimizing (14) with respect to \(\lambda\).

To prove (16), let \(0 = a(0) < a(1) < \ldots < a(n) = 1\) (recall \(K = 8n^3\)) and define the real intervals \(L(j) = [a(j), a(j + 1)]\) for \(0 \leq j \leq n - 1\) and \(L(j) = [-a(-j), -a(-j - 1)]\) for \(-n \leq j \leq -1\). Choose the \(a(j)\) such that

\[
\int f(x)^2 \, dx = (2n)^{-1}, \quad \text{for all } j.
\]

The rectangles \(\Gamma(i, j, k) = L(i) \times L(j) \times L(k)\), \(-n \leq i, j, k \leq n\) satisfy

\[
\int g(r)^2 \, d^3r = 1/K.
\]

Returning to (15), place one \(R_i\) in each of the \(K\) rectangles and then average \(W(N, R)\) over the positions, \(R_i\), of the positive particles within the rectangles with a relative weight \(g(R_1)^2 \ldots g(R_K)^2\). This average is given by

\[
W(N) = -\frac{1}{2} N^2 \sum_{m} \int_{\Gamma(m)} g(r)^2 g(r')^2
\]

\[
\times |r - r'|^{-1} \, d^3r \, d^3r',
\]

with \(m = (i, j, k)\). Since the weight is nonnegative, there is at least one choice of \(R\) (with one particle per