IV. The Minimal Surface Equation

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R. Osserman (ed.), Geometry V
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Introduction

The minimal surface equation (MSE) for functions $u: \Omega \to \mathbb{R}$, $\Omega$ a domain of $\mathbb{R}^2$, can be written

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0,$$

or equivalently $u_{xx} + u_{yy} - (1 + |Du|^2)^{-1}(u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy}) = 0$, where $u_x = \frac{\partial u(x,y)}{\partial x}$, $u_y = \frac{\partial u(x,y)}{\partial y}$. Generally, for domains $\Omega \subset \mathbb{R}^n$ and functions $u: \Omega \to \mathbb{R}$ depending on the $n$ variables $(x^1, \ldots, x^n) \in \Omega$, $n \geq 2$, the MSE can be written

$$\sum_{i,j=1}^{n} \left( \delta_{ij} - \frac{u_i u_j}{(1 + |Du|^2)^2} \right) u_{ij} = 0,$$

where $u_i = D_i u \equiv \frac{\partial u}{\partial x^i}$ and $u_{ij} = D_i D_j u$. Notice that this is a \textit{quasilinear elliptic} equation: that is, it is linear in the second derivatives, and the coefficient matrix $(\delta_{ij} - \frac{u_i u_j}{(1 + |Du|^2)^2})$ is positive definite\(^1\) depending only on the derivatives up to first order. The equation can alternatively be written in “divergence form”

$$(1) \quad \sum_{i=1}^{n} D_i \left( \frac{D_i u}{\sqrt{1 + |Du|^2}} \right) = 0,$$

which is readily checked using the chain rule and the fact that $\frac{\partial}{\partial p_j} \left( \frac{p_i}{\sqrt{1 + |p|^2}} \right) = (1 + |p|^2)^{-1/2} (\delta_{ij} - \frac{p_i p_j}{1 + |p|^2})$.

Because of its geometric significance, the MSE has historically attracted perhaps more interest than any other quasilinear elliptic equation. The geometric significance of the MSE arises from the fact that it is the Euler-Lagrange equation for the area functional

$$A(u) = \int_{\Omega} \sqrt{1 + |Du|^2},$$

representing the area ($n$-dimensional Hausdorff measure) of the graph of $u$. Thus the MSE expresses the fact that the first variation $\delta A(u)(\zeta) \equiv \frac{d}{dt} A(u + t\zeta)|_{t=0}$ vanishes for variations $u + t\zeta$ with $\zeta$ a smooth function of compact support in $\Omega$. Indeed direct computation shows that, for $\zeta \in C^1(\Omega)$ with compact support,

$$\delta A(u)(\zeta) = \int_{\Omega} \sum_{i=1}^{n} \frac{D_i u D_i \zeta}{\sqrt{1 + |Du|^2}} \equiv - \int_{\Omega} \zeta \sum_{i=1}^{n} D_i \left( \frac{D_i u}{\sqrt{1 + |Du|^2}} \right),$$

\(^1\)The coefficient matrix $(\delta_{ij} - \frac{u_i u_j}{(1 + |Du|^2)^2})$ has $(n - 1)$ eigenvalues equal to 1 and the remaining eigenvalue equal to $(1 + |Du|^2)^{-1}$; the MSE is thus a “nonuniformly elliptic” quasilinear equation, because the ratio of the maximum and minimum eigenvalues of the coefficient matrix has unbounded dependence on $Du$. 
