6. Stability and Chaos

In this chapter we study a larger class of dynamical systems that include but go beyond Hamiltonian systems. We are interested, on the one hand, in *dissipative systems*, i.e. systems that lose energy through frictional forces or into which energy is fed from exterior sources, and, on the other hand, in discrete, or discretized, systems such as those generated by studying flows by means of the Poincaré mapping. The occurrence of dissipation implies that the system is coupled to other, external systems, in a controllable manner. The strength of such couplings appears in the set of solutions, usually in the form of parameters. If these parameters are varied it may happen that the flow undergoes an essential and qualitative change, at certain critical values of the parameters. This leads rather naturally to the question of stability of the manifold of solutions against variations of the control parameters and of the nature of such a structural change. In studying these questions, one realizes that deterministic systems do not always have the well-ordered and simple behavior that we know from the integrable examples of Chap. 1, but that they may exhibit completely unordered, chaotic behavior as well. In fact, in contradiction with traditional views, and perhaps also with one’s own intuition, chaotic behavior is not restricted to dissipative systems (turbulence of viscous fluids, dynamics of climates, etc.). Even relatively simple Hamiltonian systems with a small number of degrees of freedom exhibit domains where the solutions have strongly chaotic character. As we shall see, some of these are relevant for celestial mechanics.

6.1 Qualitative Dynamics

In the preceding chapters, we dealt primarily with fundamental properties of mechanical systems, with principles that allowed the construction of their equations of motion, and with general methods of solving these equations. The integrable cases, although a minority among the dynamical systems, were of special importance because they allowed us to follow specific solutions analytically, to appreciate the significance and the power of conservation laws, and to study the restrictions that the latter impose on the manifold of motions in phase space.

On the other hand, there are questions to which we have paid less attention so far; for example: What is the long-term behavior of a periodic motion that is subject to a small perturbation? What is the structure of the flow of a mechanical system (i.e. the set of all possible solutions) in the large? Are there structural,
characteristic properties of the flow that do not depend on the specific values of the constants appearing in the equations of motion? Can there be “ordered” and “unordered” types of motions, in a given system? If yes, can one define a quantitative measure for the lack of “order”? If a given system depends on external control parameters (strength of a perturbation, amplitude and frequency of a forced vibration, varying degree of friction, etc.), are there critical values of the parameters where the flow of the system changes its structure in the large?

These questions show that, here, we approach the analysis of mechanical systems in a somewhat different spirit. The equations of motion are assumed to be known (even though they may depend on control parameters that can be varied). We concentrate less on the individual solution but, instead, study the flow as a whole, its stability, its topological structure, and its behavior over long time periods. It is this kind of analysis we wish to call qualitative dynamics. Quite logically, it leads one to investigate the stability of equilibrium positions and of periodic orbits, to study attractors for dissipative systems (i.e. manifolds of lower dimension than the original phase space, to which the system tends, for large times, under the action of dissipation), to study bifurcations (i.e. structural changes of the flow at critical values of the control parameters), and to analyse the pattern of disordered motion if it occurs.

### 6.2 Vector Fields as Dynamical Systems

The dynamics of a very great variety of dynamical systems can be cast in the form of systems of first-order differential equations, viz.

\[
\frac{d}{dt}x(t) = F(x(t), t) .
\]

Here, \(t\) is the time variable, \(x(t)\) is a point in the configuration space of the system, and \(F\) is a vector field that is continuous and often also differentiable. The space of the variables \(x\) may be the velocity space, described locally by generalized coordinates \(q^i\) and velocities \(\dot{q}^i\), or the phase space that we describe locally by the \(q^i\) and the canonically conjugate momenta \(p_i\). There are, of course, other cases where the \(x\) live in some other manifold: an example is provided by the Eulerian angles that parametrize the rotational motion of rigid bodies.

As an example, let the equation of the motion be given in the form

\[
\ddot{y} + f_1(y, t)\dot{y} + f_2(y, t) = 0 .
\]

It is easy to recast this in the form of (6.1), by taking

\[
x_1(t) \overset{\text{def}}{=} y(t) , \quad x_2(t) \overset{\text{def}}{=} \dot{y}(t) ,
\]

so that \(\dot{x}_1 = x_2, \dot{x}_2 = -f_1x_2 - f_2\). The pattern (6.1), of course, is not restricted to Lagrangian or Hamiltonian systems. It also describes systems with dissipation,