Chapter III

Riemann-Roch Theory

§ 1. Primes

Having set up the general theory of valued fields, we now return to algebraic number fields. We want to develop their basic theory from the valuation-theoretic point of view. This approach has two prominent advantages compared to the ideal-theoretic treatment given in the first chapter. The first one results from the possibility of passing to completions, thereby enacting the important "local-to-global principle". This will be done in chapter VI. The other advantage lies in the fact that the archimedean valuations bring into the picture the points at infinity, which were hitherto lacking, as the "primes at infinity". In this way a perfect analogy with the function fields is achieved. This is the task to which we now turn.

(1.1) Definition. A prime (or place) \( p \) of an algebraic number field \( K \) is a class of equivalent valuations of \( K \). The nonarchimedean equivalence classes are called finite primes and the archimedean ones infinite primes.

The infinite primes \( p \) are obtained, according to chap. II, (8.1), from the embeddings \( \tau : K \rightarrow \mathbb{C} \). There are two sorts of these: the real primes, which are given by embeddings \( \tau : K \rightarrow \mathbb{R} \), and the complex primes, which are induced by the pairs of complex conjugate non-real embeddings \( K \rightarrow \mathbb{C} \). \( p \) is real or complex depending whether the completion \( K_p \) is isomorphic to \( \mathbb{R} \) or to \( \mathbb{C} \). The infinite primes will be referred to by the formal notation \( p \mid \infty \), the finite ones by \( p \nmid \infty \).

In the case of a finite prime, the letter \( p \) has a multiple meaning: it also stands for the prime ideal of the ring \( \mathfrak{o} \) of integers of \( K \), or for the maximal ideal of the associated valuation ring, or even for the maximal ideal of the completion. However, this will nowhere create any risk of confusion. We write \( p \mid p \) if \( p \) is the characteristic of the residue field \( \kappa(p) \) of the finite prime \( p \). For an infinite prime we adopt the convention that the completion \( K_p \) also serves as its own "residue field," i.e., we put

\[
\kappa(p) := K_p, \quad \text{when } p \mid \infty. \]
To each prime $p$ of $K$ we now associate a canonical homomorphism 
\[ v_p : K^* \rightarrow \mathbb{R} \]
from the multiplicative group $K^*$ of $K$. If $p$ is finite, then $v_p$ is the $p$-adic exponential valuation which is normalized by the condition $v_p(K^*) = \mathbb{Z}$. If $p$ is infinite, then $v_p$ is given by
\[ v_p(a) = -\log |\tau a|, \]
where $\tau : K \rightarrow \mathbb{C}$ is an embedding which defines $p$.

For an arbitrary prime $p|p$ ($p$ prime number or $p = \infty$) we put furthermore
\[ f_p = \left[ \kappa(p) : \kappa(p) \right], \]
so that $f_p = [K_p : \mathbb{R}]$ if $p|\infty$, and
\[ \mathfrak{N}(p) = \begin{cases} p^{f_p}, & \text{if } p \not| \infty, \\ e^{f_p}, & \text{if } p | \infty. \end{cases} \]

This convention suggests that we consider $e$ as being an infinite prime number, and the extension $\mathbb{C} | \mathbb{R}$ as being unramified with inertia degree $2$.

We define the absolute value $| \cdot |_p : K \rightarrow \mathbb{R}$ by
\[ |a|_p = \mathfrak{N}(p)^{-v_p(a)} \]
for $a \neq 0$ and $|0|_p = 0$. If $p$ is the infinite prime associated to the embedding $\tau : K \rightarrow \mathbb{C}$, then one finds
\[ |a|_p = |\tau a|, \quad \text{resp. } |a|_p = |\tau a|^2, \]
if $p$ is real, resp. complex.

If $L | K$ is a finite extension of $K$, then we denote the primes of $L$ by $\mathfrak{P}$ and write $\mathfrak{P}|p$ to signify that the valuations in the class $\mathfrak{P}$, when restricted to $K$, give those of $p$. In the case of an infinite prime $\mathfrak{P}$, we define the inertia degree, resp. the ramification index, by
\[ f_{\mathfrak{P}|p} = [L_{\mathfrak{P}} : K_p], \quad \text{resp. } e_{\mathfrak{P}|p} = 1. \]

For arbitrary primes $\mathfrak{P}|p$ we then have the

(1.2) Proposition. \begin{enumerate}
\item \[ \sum_{\mathfrak{P}|p} e_{\mathfrak{P}|p} f_{\mathfrak{P}|p} = \sum_{\mathfrak{P}|p} [L_{\mathfrak{P}} : K_p] = [L : K], \]
\item \[ \mathfrak{N}(\mathfrak{P}) = \mathfrak{N}(p)^{f_{\mathfrak{P}|p}}, \]
\item \[ v_\mathfrak{P}(a) = e_{\mathfrak{P}|p} v_p(a) \quad \text{for } a \in K^*, \]
\item \[ v_p(N_{L_{\mathfrak{P}}|K_p}(a)) = f_{\mathfrak{P}|p} v_\mathfrak{P}(a) \quad \text{for } a \in L^*, \]
\item \[ |a|_\mathfrak{P} = |N_{L_{\mathfrak{P}}|K_p}(a)|_p \quad \text{for } a \in L. \]
\end{enumerate}