6 General Solutions of the Heat Equation

6.1 The Boundary-Value Problem

The substrate shall be an infinite slab of uniform thickness, $h_s$, that is irradiated by a cw- or pulsed-laser beam which is either focused or extended over a wider area (Fig. 6.1.1). For localized irradiation, the absorbed laser light generates a local temperature rise, $\Delta T(x, t)$, which can be calculated by solving the three-dimensional heat equation (2.2.1). For large-area (uniform) irradiation, the temperature is uniform within planes $z = \text{const.}$, and the temperature rise, $\Delta T(z, t)$, can be calculated by solving the one-dimensional heat equation.

![Fig. 6.1.1. Infinite slab of uniform thickness, $h_s$, irradiated by a laser beam at perpendicular incidence. In the absence of scanning, the origin of the coordinate system is on the surface in the center of the laser beam. If $v_s \neq 0$, the origin of the coordinate system is either fixed with the substrate or with the laser beam, depending on the particular problem under consideration. The laser-induced temperature rise on the surface $z = 0$ and along the $z$-direction is indicated (dotted curves). The temperature far away from the irradiated area is $T(\infty)$. With increasing focal width, $2w$, the temperature distribution becomes wider. In the limit $w \to \infty$ (large-area irradiation) the temperature rise becomes uniform within planes $z = \text{const.}$](image-url)
The Source Term

In most cases of thermal laser processing, the Rayleigh length is long compared to the optical penetration depth. Then, in a coordinate system that is fixed with the laser beam, the source term in the heat equation can be written in the form

\[ Q(x_a, t) = I(x, y, t)(1 - R)f(z) = I_a g(x, y) f(z) q(t) , \]  

(6.1.1)

where \( I_a = I_0 (1 - R) \) is the (maximum) laser-light intensity that is not reflected from the surface \( z = 0 \). \( g(x, y) \) describes the (arbitrary) intensity distribution (beam shape) within the \( xy \)-plane. \( f(z) \) describes the attenuation of the laser light in \( z \)-direction, and \( q(t) \) its temporal dependence (pulse shape). Thus, we have \( \max[g(x, y)] = \max[q(t)] = 1 \). Frequently, we introduce cylindrical coordinates, so that \( I = I(r, \varphi, t) \), where \( \varphi \) describes the angle between the radius vector \( r \) and the \( x \)-axis.

\( R = R(T, \lambda) \) denotes the temperature- and wavelength-dependent normal-incidence reflectivity within the processed area. In the general case, the reflectivity depends on the angle of incidence, the polarization of the laser beam, and the thickness of the slab. The latter dependence can be ignored if the optical penetration depth \( l_o \ll h_s \). If, on the other hand, \( l_o > h_s \), interference phenomena due to multiple reflections of the laser light may become important. The reflectivity, and also the absorptivity, will then depend on \( h_s \). In this chapter we ignore multiple reflections within the slab.

Values of \( R \) and \( \alpha \) are listed in Table III for different materials and various wavelengths. For metals, reflectivity values in the near UV and VIS spectral range are, typically, between 0.4 and 0.95. In the IR, typical values are between 0.9 and 0.99. It should be noted, however, that these values can be applied only if the wavelength, \( \lambda \), is small compared to the radius of the laser spot, \( w \). Additionally, in the dependence \( R = R(T, \lambda) \) we can employ the surface temperature \( T = T(x, y, 0, t) \) only if the variation of \( T = T(x, t) \) in \( z \)-direction is slow over the distance \( l_o \), i.e., if \( l_o (dT/dz) \ll T \). Finally, with very high laser-light intensities, optical non-linearities become important.

6.1.1 The Attenuation Function, \( f(z) \)

The function \( f(z) \) in (6.1.1) describes the attenuation of the laser beam in \( z \)-direction. For a uniform material, it can be written as

\[ f(z) = \alpha(T(z)) \exp \left[ - \int_0^z \alpha(T(z')) dz' \right] , \]  

(6.1.2)

where \( \alpha(T) \) is the temperature-dependent linear absorption coefficient at the laser wavelength under consideration. Because of this temperature dependence, the optical properties of the material become inhomogeneous under laser-light irradiation. This case is discussed in Chap. 9. In the present chapter