In this chapter, we show that depending on the coefficients, the complex Riccati equation has one or another homogeneity domain of the space of several complex variables as its integral manifold. A domain $D \subset \mathbb{C}^n$ is called a homogeneity domain if there exists an infinite group of analytic automorphisms of this domain onto itself. Homogeneity domains (Cartan–Siegel domains) often occur in many important branches of calculus (for example, in analytic number theory, automorphic function theory, etc. [16, 50, 82, 99]). We present the main facts related to the homogeneity domains.

§1. Cartan–Siegel Domains

Homogeneity domains first appeared in the studies by E. Cartan [16] as domains $D$ of the space $\mathbb{C}^n$ that possess an infinite group $G$ of analytic automorphisms.

For $n = 1$, the unit disk $|z| < 1$ is a unique (up to analytic isomorphisms) bounded homogeneity domain. The group $G$ of its analytic automorphisms consists of linear-fractional transformations

$$z \mapsto \frac{(az + b)}{(bz + a)},$$

where $a$ and $b$ are complex numbers such that $a\bar{a} - b\bar{b} = 1$.

For $n \geq 2$, the description of all homogeneity domains is a difficult question. The answer becomes comprehensible under the additional assumption that the group $G$ acts transitively on the domain $D$ (i.e., $G$ contains transformations that transform any point of $D$ into another point of $D$). The group $G$ of all analytic automorphisms of a bounded domain $D \subset \mathbb{C}^n$ turns out to be a real Lie group (see [16], p. 134), and the domain $D$ can therefore be described as a quotient space of the Lie group $G$ by the stationary subgroup. Therefore, the domain $D$ becomes an (analytic) homogeneous space. Because of this, such a domain is called a homogeneity domain of the space of several complex variables.

We note that to describe all homogeneity domains, it suffices to find only one domain from each equivalence class of domains (with respect to analytic mappings); moreover, it suffices to consider only irreducible classes of homogeneity domains, i.e., the classes that are not direct products of homogeneity domains of lower dimensions.

E. Cartan showed (see [16]) that for $n = 2$, any bounded irreducible homogeneity domain is equivalent to the domain $|z_1|^2 + |z_2|^2 < 1$. Moreover, he
§1. Cartan–Siegel Domains

conjectured that for $n = 3$, any bounded irreducible homogeneity domain is equivalent to one of the following two domains:

$$|z_1|^2 + |z_2|^2 + |z_3|^2 < 1$$

or

$$Z\overline{Z} < I_2, \quad Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$  

For $n \geq 3$, E. Cartan introduced an additional requirement. Namely, he assumed that the domain $D$ is symmetrical in the sense that for each point $z \in D$, the group $G$ contains an involution with a unique fixed point $z$. (For $n \leq 3$, this assumption holds automatically; see [16].) This assumption allows a classification of all bounded symmetrical domains $D$.

Remark. In Chap. 4, we considered Riemannian symmetrical spaces corresponding to compact Lie groups. Symmetrical homogeneity domains, Hermitian symmetrical spaces, correspond to noncompact Lie groups. (For example, the unit disk $|z| < 1$ corresponds to the noncompact Lie group of its analytic automorphisms. Indeed, this group coincides with $\text{SL}(2, \mathbb{R})$ because the unit disk is analytically isomorphic to the upper half-plane [96]).

Bounded nonsymmetrical homogeneity domains were first described by I. I. Pyatetskii–Shapiro in [83].

Among symmetrical homogeneous domains, we find four types of irreducible domains. Moreover, there is an exceptional domain in each of the spaces $\mathbb{C}^{16}$ and $\mathbb{C}^{27}$, which we do not describe here. Further, when describing these four types of irreducible bounded symmetrical domains, we follow the Siegel presentation [99] (with certain modifications).

Type I. Let $p$ and $q$ be two positive integers, $p + q = m$. We consider the group of complex linear transformations $G_1 = U(p, q)$ of the space $\mathbb{C}^m$ that leave the invariant Hermitian form with $p$ negative and $q$ positive squares invariant:

$$\Omega(t) = -|t_1|^2 - \cdots - |t_p|^2 + |t_{p+1}|^2 + \cdots + |t_m|^2 = t^T H \bar{t},$$

where

$$t = \begin{pmatrix} t_1 \\ \vdots \\ t_m \end{pmatrix}, \quad H = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}.$$  

In other words, we consider the linear transformations with the matrix $g \in \mathcal{M}_m(\mathbb{C})$ satisfying the relations $g^T H \bar{g} = H$. We agree to represent the number of rows and columns of the matrix with double superscripts (the first is the number of rows, and the second is the number of columns). We set