Chapter 13
Harmonic Functionals and Related Topics

§1. Riemannian Volume Element

In a (pseudo-)Riemannian space $\mathcal{X}$, we can measure not only lengths but also volumes, i.e., we can define a natural volume density $dV$ on $\mathcal{X}$. Indeed, let $(U, h)$ be an arbitrary chart of an arbitrary (pseudo-)Riemannian space $\mathcal{X}$, let $[g_{ij}]$ be the matrix of components of the metric tensor $g$ in the chart $(U, h)$, and let

$$\det g = |g_{ij}|$$

be its determinant. The transformation formula for the matrix of a quadratic form under a change of basis directly implies that under a change of coordinates, the determinant $\det g$ is multiplied by the square of the Jacobian of the transition functions. Therefore, setting

$$dV^U = \sqrt{\det g}$$

(as usual, we mean the arithmetical square root), we obtain a certain volume density $dV$ on $\mathcal{X}$. (Of course, for a Riemannian space, it is not necessary to pass from $\det g$ to $|\det g|$.) The density $dV$ is conventionally denoted by

$$\sqrt{|\det g|} \, dx^1 \cdots dx^n$$

or

$$\sqrt{|\det g|} \, dx.$$  \hspace{1cm} (1)

**Definition 13.1.** Density (1) is called the **Riemannian volume element** on the (pseudo-)Riemannian manifold $\mathcal{X}$.

For $n = 2$, this is the already known area element $\sqrt{EG - F^2} \, du \, dv$ on a surface.

If a manifold $\mathcal{X}$ is orientable and oriented, then we can pass from the density $dV$ to the corresponding form of maximal degree. This form is also called the **volume element** (sometimes the **oriented volume element**) and is denoted by the previous symbol $dV$. By definition,

$$dV = \sqrt{|\det g|} \, dx^1 \wedge \cdots \wedge dx^n$$

in each positively oriented chart $(U, x^1, \ldots, x^n)$.

§2. Discriminant Tensor

The coefficients of the form $dV$ compose a tensor field of type $(n, 0)$ on $\mathcal{X}$. This field is called the **discriminant tensor** of a (pseudo-)Riemannian manifold.
\( X \) and is traditionally denoted by \( e \) (the symbol \( \varepsilon \) is also used); its components are denoted by \( e_{i_1 \cdots i_n} \) (or \( \varepsilon_{i_1 \cdots i_n} \)). Therefore, in each chart \((U, h)\), we have

\[
e_{i_1 \cdots i_n} = \begin{cases} \varepsilon_\sigma e_{1 \cdots n} & \text{if the subscripts } i_1, \ldots, i_n \text{ are distinct,} \\ 0 & \text{otherwise,} \end{cases}
\]

where \( \varepsilon_\sigma \) is the sign of the permutation

\[
\sigma = \begin{pmatrix} 1 & \cdots & n \\ i_1 & \cdots & i_n \end{pmatrix}
\]

and

\[
e_{1 \cdots n} = \sqrt{|\det g|}
\]

if the orientation of the chart \((U, h)\) is positive and

\[
e_{1 \cdots n} = -\sqrt{|\det g|}
\]

if the orientation is negative.

For example, for \( n = 2 \),

\[
\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} = \begin{bmatrix} 0 & \pm \sqrt{EG - F^2} \\ \mp \sqrt{EG - F^2} & 0 \end{bmatrix},
\]

where the signs \( \pm \) depend on the orientation of the chart.

\section*{§3. Foss–Weyl Formula}

The linear differential form

\[
\gamma = \Gamma_{ik}^i \, dx^k
\]

(see Chap. 2) is closely related to the discriminant tensor.

**Proposition 13.1.** The formula

\[
\gamma = d \ln \sqrt{|\det g|}
\]

holds.

**Proof.** Formula \((5')\) in Chap. 11 implies

\[
\Gamma_{ik}^i = \frac{1}{2} g^{ip} \left( \frac{\partial g_{pi}}{\partial x^k} + \frac{\partial g_{pk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^p} \right) = \frac{1}{2} g_{ij} \frac{\partial g_{ij}}{\partial x^k},
\]

i.e.,

\[
\gamma = \frac{1}{2} g_{ij} \, dg_{ij} = \frac{1}{2} \text{Tr}(g^{-1} \, dg),
\]

where \( g \) is the matrix \( \|g_{ij}\| \). On the other hand,