Chapter 3
Affine Mappings. Submanifolds

§1. Affine Mappings

Let $\mathcal{X}$ and $\mathcal{Y}$ be affine connection spaces with the connections $\nabla^\mathcal{X}$ and $\nabla^\mathcal{Y}$ (to simplify formulas, we often write $\nabla$ instead of $\nabla^\mathcal{X}$ and $\nabla^\mathcal{Y}$). On each coordinate neighborhood $U$ of the manifold $\mathcal{X}$ (coordinate neighborhood $V$ of the manifold $\mathcal{Y}$), the connection $\nabla^\mathcal{X}$ (connection $\nabla^\mathcal{Y}$) is given by the matrix $\omega = \omega^\mathcal{X}$ (matrix $\bar{\omega} = \omega^\mathcal{Y}$) of connection forms. The connection $\nabla^\mathcal{X}$ sets the horizontal subspace $H^A_\mathcal{X}$ of the tangent space $T_A(\mathcal{T}\mathcal{X})$ in correspondence with each tangent vector $A$ (point of the total space $\mathcal{T}\mathcal{X}$ of the tangent bundle $\tau\mathcal{X}$). Similarly, the connection $\nabla^\mathcal{Y}$ sets the horizontal subspace $H^B_\mathcal{Y} \subset T_B(\mathcal{T}\mathcal{Y})$ in correspondence with each point $B \in \mathcal{T}\mathcal{Y}$.

Let $f: \mathcal{X} \to \mathcal{Y}$ be an arbitrary smooth mapping. Two charts $(U, h) = (U, x^1, \ldots, x^n)$ and $(V, k) = (V, y^1, \ldots, y^m)$ of manifolds $\mathcal{X}$ and $\mathcal{Y}$ are said to be $f$-related if $fU \subset V$. In such charts, the mapping $f$ (or, more precisely, the mapping $U \to V$ induced by it) is given by functions of the form

$$ y^a = f^a(x^1, \ldots, x^n), \quad a = 1, \ldots, m. $$

The Jacobi matrix

$$ J_f = \left\| \frac{\partial f^a}{\partial x^i} \right\|, \quad 1 \leq i \leq n, \quad 1 \leq a \leq m, $$

of these functions is called the Jacobi matrix of the mapping $f$ in the charts $(U, h)$ and $(V, k)$.

We recall that the vector fields $X \in \mathfrak{a}\mathcal{X}$ and $\hat{X} \in \mathfrak{a}\mathcal{Y}$ are said to be $f$-related if

$$ (df)_p X_p = \hat{X}_{f(p)} \quad (1) $$

for any point $p \in \mathcal{X}$, i.e., if for any pair of $f$-related charts $(U, h)$ and $(V, k)$, we have

$$ \frac{\partial f^a}{\partial x^i} X^i = \hat{X}^a \circ f, \quad 1 \leq i \leq n, \quad 1 \leq a \leq m, $$

where $X^i$ and $\hat{X}^a$ are components if the fields $X$ and $\hat{X}$ in the charts $(U, h)$ and $(V, k)$.

On the other hand, it is clear that the formula

$$ (\mathcal{T}f)A = (df)_p A, \quad A \in \mathcal{T}\mathcal{X}, $$

where $p \in \mathcal{X}$ is a point such that $A \in \mathcal{T}_p\mathcal{X}$, correctly defines the smooth mapping

$$ \mathcal{T}f: \mathcal{T}\mathcal{X} \to \mathcal{T}\mathcal{Y} $$
of manifolds of tangent vectors. Because this mapping is smooth, its differential at an arbitrary point \( A \in \mathbf{T}X \)

\[
(dTf)_A : \mathbf{T}_A(\mathbf{T}X) \to \mathbf{T}_B(\mathbf{T}Y)
\]

is defined, where \( B = (Tf)_A \).

**Proposition 3.1.** The following properties of a smooth mapping \( f : \mathcal{X} \to \mathcal{Y} \) are equivalent:

1. If two fields \( X, Y \in \mathfrak{aX} \) are \( f \)-related to the fields \( \hat{X}, \hat{Y} \in \mathfrak{aY} \), then the field \( \nabla_X Y \) is \( f \)-related to the field \( \hat{\nabla}_{\hat{X}} \hat{Y} \).

2. For any \( f \)-related charts \( (U, h) \) and \( (\hat{U}, \hat{h}) \),

\[
J_f \omega = f^* \hat{\omega} J_f + dJ_f \quad \text{on } U. \tag{2}
\]

3. For any curve \( \gamma : I \to \mathcal{X} \) and any vector field \( X : t \mapsto X(t) \) on \( \gamma \),

\[
\frac{d}{dt}[(df)_{\gamma(t)} X(t)] = (df)_{\gamma(t)} \nabla_{\gamma(t)} X(t), \quad t \in I. \tag{3}
\]

4. For any curve \( \gamma : I \to \mathcal{X} \), the diagram

\[
\begin{array}{ccc}
\mathbf{T}_{p_0} \mathcal{X} & \xrightarrow{(df)_{p_0}} & \mathbf{T}_{q_0} \mathcal{Y} \\
\Pi_{\gamma} \downarrow & & \downarrow \hat{\Pi}_{f_{\gamma}} \\
\mathbf{T}_{p_1} \mathcal{X} & \xrightarrow{(df)_{p_1}} & \mathbf{T}_{q_1} \mathcal{Y}
\end{array}
\]

is commutative, where \( p_0 \) and \( p_1 \) are the respective initial point and endpoint of the curve \( \gamma \), \( q_0 = f(p_0) \), \( q_1 = f(p_1) \), and \( \Pi_{\gamma} \) and \( \hat{\Pi}_{f_{\gamma}} \) are parallel translations along the curves \( \gamma \) and \( f \circ \gamma \).

5. At each point \( A \in \mathbf{T}X \),

\[
(dTf)_A H^X_A \subset H^Y_B, \quad B = (Tf)_A.
\]

**Proof.** If the fields \( X, Y \in \mathfrak{aX} \) and \( \hat{X}, \hat{Y} \in \mathfrak{aY} \) are \( f \)-related, then for any \( f \)-related charts \( (U, h) \) and \( (\hat{V}, k) \),

\[
X^i \frac{\partial f^a}{\partial x^i} = \hat{X}^a \circ f, \quad Y^i \frac{\partial f^a}{\partial x^i} = \hat{Y}^a \circ f \quad \text{on } U.
\]

On the other hand, if \( \Gamma^i_{kj} \) and \( \hat{\Gamma}^a_{cb} \) are coefficients of the connections \( \nabla \) and \( \hat{\nabla} \) in these charts, then

\[
(\nabla_X Y)^i = \left( \frac{\partial Y^i}{\partial x^k} + \Gamma^i_{kj} Y^j \right) X^k, \quad (\hat{\nabla}_{\hat{X}} \hat{Y})^a = \left( \frac{\partial \hat{Y}^a}{\partial y^c} + \hat{\Gamma}^a_{cb} \hat{Y}^b \right) \hat{X}^c.
\]

Therefore,