The multivariate regionalization of a set of random functions can be represented with a spatial multivariate linear model. The associated multivariate nested variogram model is easily fitted to the multivariate data. Several coregionalization matrices describing the multivariate correlation structure at different scales of a phenomenon result from the variogram fit. The relation between the coregionalization matrices and the classical variance-covariance matrix is examined.

**Linear model of coregionalization**

A set of real second-order stationary random functions \( \{Z_i(x); i = 1, \ldots, N\} \) can be decomposed into sets \( \{Z^u_i(x); u = 0, \ldots, S\} \) of spatially uncorrelated components

\[
Z_i(x) = \sum_{u=0}^{S} Z^u_i(x) + m_i, \tag{26.1}
\]

where for all values of the indices \( i, j, u \) and \( v \),

\[
E[Z_i(x)] = m_i, \tag{26.2}
\]

\[
E[Z^u_i(x)] = 0, \tag{26.3}
\]

and

\[
\text{cov}(Z^u_i(x), Z^v_j(x+h)) = E[Z^u_i(x)Z^v_j(x+h)] = C_{ij}^u(h), \tag{26.4}
\]

\[
\text{cov}(Z^u_i(x), Z^u_j(x+h)) = 0 \quad \text{when} \quad u \neq v. \tag{26.5}
\]

The cross covariance functions \( C_{ij}^u(h) \) associated with the spatial components are composed of real coefficients \( b^u_{ij} \) and are proportional to real correlation functions \( \rho_u(h) \)

\[
C_{ij}(h) = \sum_{u=0}^{S} C_{ij}^u(h) = \sum_{u=0}^{S} b^u_{ij} \rho_u(h), \tag{26.6}
\]

which implies that the cross covariance functions are even in this model.

Coregionalization matrices \( B_u \) of order \( N \times N \) can be set up and we have a multivariate nested covariance function model

\[
C(h) = \sum_{u=0}^{S} B_u \rho_u(h) \tag{26.7}
\]
with positive semi-definite coregionalization matrices $B_u$.

EXERCISE 26.1 When is the above covariance function model equivalent to the intrinsic correlation model?

EXERCISE 26.2 Show that a correlation function $\rho_u(h)$ having a non zero sill $b_{ij}$ on a given cross covariance function has necessarily non zero sills $b_{ii}$ and $b_{jj}$ on the corresponding direct covariance functions.

Conversely, if a sill $b_{ii}$ is zero for a given structure of a variable, all sills of the structure on all cross covariance functions with this variable are zero.

Each spatial component $Z_u^i(x)$ can itself be represented as a set of uncorrelated factors $Y_u^p(x)$ with transformation coefficients $a_{ip}$,

$$Z_u^i(x) = \sum_{p=1}^{N} a_{ip} Y_u^p(x), \quad (26.8)$$

where for all values of the indices $i, j, u, v, p$ and $q$,

$$E[Y_u^p(x)] = 0, \quad (26.9)$$

and,

$$\text{cov}(Y_u^p(x), Y_u^p(x+h)) = \rho_u(h), \quad (26.10)$$

$$\text{cov}(Y_u^p(x), Y_u^q(x+h)) = 0 \quad \text{when} \quad u \neq v \quad \text{or} \quad p \neq q. \quad (26.11)$$

Combining the spatial with the multivariate decomposition, we obtain the linear model of coregionalization

$$Z_i(x) = \sum_{u=0}^{S} \sum_{p=1}^{N} a_{ip}^u Y_u^p(x). \quad (26.12)$$

In practice first a set of correlation functions $\rho_u(h)$ (i.e. normalized variograms $g_u(h)$) is selected, taking care to keep $S$ reasonably small. Then the coregionalization matrices are fitted using a weighted least squares algorithm (described below). The weighting coefficients are chosen by the practitioner so as to provide a graphically satisfactory fit which downweighs arbitrarily distance classes which do not comply with the shape suggested by the experimental variograms. Finally the coregionalization matrices are decomposed, yielding the transformation coefficients $a_{ip}^u$, which specify the linear coregionalization model

$$B_u = A_u A_u^T \quad \text{where} \quad A_u = [a_{ip}^u]. \quad (26.13)$$

The decomposition of the $B_u$ into the product of $A_u$ with its transpose is usually based on the eigenvalue decomposition of each coregionalization matrix. Several decompositions for the purpose of a regionalized multivariate data analysis are discussed in the next chapter.