17 Dynamical Systems and Chaos

17.1 Ordinary Differential Equations and Mappings

17.1.1 Dynamical Systems

17.1.1.1 Basic Notions

1. The Notion of Dynamical Systems and Orbits

A dynamical system is a mathematical object to describe the development of a physical, biological or another system from real life depending on time. It is defined by a phase space $M$, and by a one-parameter family of mappings $\varphi^t : M \to M$, where $t$ is the parameter (the time). In the following, the phase space is often $\mathbb{R}^n$, a subset of it, or a metric space. The time parameter $t$ is from $\mathbb{R}$ (time continuous system) or from $\mathbb{Z}$ or from $\mathbb{Z}^+$ (time discrete system). Furthermore, it is required for arbitrary $x \in M$ that

a) $\varphi^0(x) = x$ and

b) $\varphi^s(\varphi^t(x)) = \varphi^{s+t}(x)$ for all $t, s$. The mapping $\varphi^t$ is denoted briefly by $\varphi$.

In the following, the time set is denoted by $\Gamma$, hence, $\Gamma = \mathbb{R}, \Gamma = \mathbb{R}_+, \Gamma = \mathbb{Z}$ or $\Gamma = \mathbb{Z}_+$. If $\Gamma = \mathbb{R}$, then the dynamical system is also called a flow; if $\Gamma = \mathbb{Z}$ or $\Gamma = \mathbb{Z}_+$, then the dynamical system is discrete. In case $\Gamma = \mathbb{R}$ and $\Gamma = \mathbb{Z}$, the properties a) and b) are satisfied for every $t \in \Gamma$, so the inverse mapping $(\varphi^t)^{-1} = \varphi^{-t}$ also exists, and these systems are called invertible dynamical systems.

If the dynamical system is not invertible, then $\varphi^{-t}(A)$ means the pre-image of $A$ with respect to $\varphi^t$, for an arbitrary set $A \subset M$ and arbitrary $t > 0$, i.e., $\varphi^{-t}(A) = \{x \in M : \varphi^t(x) \in A\}$. If the mapping $\varphi^t : M \to M$ is continuous or $k$ times continuously differentiable for every $t \in \Gamma$ (here $M \subset \mathbb{R}^n$), then the dynamical system is called $C^k$-smooth, respectively.

For an arbitrary fixed $x \in M$, the mapping $t \mapsto \varphi^t(x), t \in \Gamma$, defines a motion of the dynamical system starting from $x$ at time $t = 0$. The image $\gamma(x)$ of a motion starting at $x$ is called the orbit (or the trajectory) through $x$, namely $\gamma(x) = \{\varphi^t(x)\}_{t \in \Gamma}$. Analogously, the positive semiorbit through $x$ is defined by $\gamma^+(x) = \{\varphi^t(x)\}_{t \geq 0}$ and, if $\Gamma \neq \mathbb{R}_+$ or $\Gamma \neq \mathbb{Z}_+$, then the negative semiorbit through $x$ is defined by $\gamma^-(x) = \{\varphi^t(x)\}_{t \leq 0}$.

The orbit $\gamma(x)$ is a steady state (also equilibrium point or stationary point) if $\gamma(x) = \{x\}$, and it is $T$-periodic if there exists a $T \in \Gamma, T > 0$, such that $\varphi^{t+T}(x) = \varphi^t(x)$ for all $t \in \Gamma$, and $T \in \Gamma$ is the smallest positive number with this property. The number $T$ is called the period.

2. Flow of a Differential Equation

Consider the ordinary linear planar differential equation

\[ \dot{x} = f(x), \quad (17.1) \]

where $f : M \to \mathbb{R}^n$ (vector field) is an $r$-times continuously differentiable mapping and $M = \mathbb{R}^n$ or $M$ is an open subset of $\mathbb{R}^n$. In the following, the Euclidean norm $\|x\|$ is used in $\mathbb{R}^n$, i.e., for arbitrary $x \in \mathbb{R}^n, x = (x_1, \ldots, x_n)$, its norm is $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$. If the mapping $f$ is written componentwise

\[ f = (f_1, \ldots, f_n), \]

then (17.1) is a system of $n$ scalar differential equations $\dot{x}_i = f_i(x_1, \ldots, x_n), \; i = 1, 2, \ldots, n$.

The Picard–Lindelöf theorem on the local existence and uniqueness of solutions of differential equations locally and the theorem on the $r$-times differentiability of solutions with respect to the initial values (see [17.11]) guarantee that for every $x_0 \in M$, there exist a number $\varepsilon > 0$, a sphere $B_\delta(x_0) = \{x : \|x - x_0\| < \delta\}$ in $M$ and a mapping $\varphi : (-\varepsilon, \varepsilon) \times B_\delta(x_0) \to M$ such that:

1. $\varphi(\cdot, \cdot)$ is $(r + 1)$-times continuously differentiable with respect to its first argument (time) and $r$-times continuously differentiable with respect to its second argument (phase variable);
2. for every fixed \( x \in B_\delta(x_0) \), \( \varphi(\cdot, x) \) is the locally unique solution of (17.1) in the time interval \((-\varepsilon, \varepsilon)\) which starts from \( x \) at time \( t = 0 \), i.e., \( \frac{\partial \varphi}{\partial t}(t, x) = \varphi(t, x) = f(\varphi(t, x)) \) holds for every \( t \in (-\varepsilon, \varepsilon) \), \( \varphi(0, x) = x \), and every other solution with initial point \( x \) at time \( t = 0 \) coincides with \( \varphi(t, x) \) for all small \(|t|\).

Suppose that every local solution of (17.1) can be extended uniquely to the whole of \( \mathbb{R} \). Then there exists a mapping \( \varphi: \mathbb{R} \times M \to M \) with the following properties:

1. \( \varphi(0, x) = x \) for all \( x \in M \).
2. \( \varphi(t + s, x) = \varphi(t, \varphi(s, x)) \) for all \( t, s \in \mathbb{R} \) and all \( x \in M \).
3. \( \varphi(\cdot, \cdot) \) is continuously differentiable \((r + 1)\) times with respect to its first argument and \( r \) times with respect to the second one.
4. For every fixed \( x \in M \), \( \varphi(\cdot, x) \) is a solution of (17.1) on the whole of \( \mathbb{R} \). Then the \( C^r \)-smooth flow generated by (17.1) can be defined by \( \varphi^t = \varphi(t, \cdot) \). The motions \( \varphi(\cdot, x): \mathbb{R} \to M \) of a flow of (17.1) are called integral curves.

The equation
\[
\dot{x} = \sigma(y - x), \quad \dot{y} = rx - y - xz, \quad \dot{z} = xy - bz
\] (17.2)
is called a Lorenz system of convective turbulence (see also 17.2.4.3, p. 825). Here \( \sigma > 0 \), \( r > 0 \) and \( b > 0 \) are parameters. The Lorenz system corresponds to a \( C^\infty \) flow on \( M = \mathbb{R}^3 \).

3. Discrete Dynamical System

Consider the difference equation
\[
x_{t+1} = \varphi(x_t),
\] (17.3)
which can also be written as an assignment \( x \mapsto \varphi(x) \). Here \( \varphi: M \to M \) is a continuous or \( r \) times continuously differentiable mapping, where in the second case \( M \subset \mathbb{R}^n \). If \( \varphi \) is invertible, then (17.3) defines an invertible discrete dynamical system through the iteration of \( \varphi \), namely,
\[
\varphi^t = \underbrace{\varphi \circ \cdots \circ \varphi}_{t \text{ times}}, \quad \varphi^t = \varphi^{-1} \circ \cdots \circ \varphi^{-1}, \quad \text{for} \quad t < 0, \quad \varphi^0 = \text{id}.
\] (17.4)
If \( \varphi \) is not invertible, then the mappings \( \varphi^t \) are defined only for \( t \geq 0 \). For the realization of \( \varphi^t \) see (5.86), p. 296.

A: The difference equation
\[
x_{t+1} = ax_t (1 - x_t), \quad t = 0, 1, \ldots \tag{17.5}
\]
with parameter \( a \in (0, 4] \) is called a logistic equation. Here \( M = [0, 1] \), and \( \varphi: [0, 1] \to [0, 1] \) is, for a fixed \( a \), the function \( \varphi(x) = ax(1 - x) \). Obviously, \( \varphi \) is infinitely many times differentiable, but not invertible. Hence (17.5) defines a non-invertible dynamical system.

B: The difference equation
\[
x_{t+1} = y_t + 1 - ax_t^2, \quad y_{t+1} = bx_t, \quad t = 0, \pm 1, \ldots \tag{17.6}
\]
with parameters \( a > 0 \) and \( b \neq 0 \) is called a Hénon mapping. The mapping \( \varphi: \mathbb{R}^2 \to \mathbb{R}^2 \) corresponding to (17.6) is defined by \( \varphi(x, y) = (y + 1 - ax^2, bx) \), is infinitely often differentiable and invertible.

4. Volume Contracting and Volume Preserving Systems

The invertible dynamical system \( \{\varphi^t\}_{t \in \mathbb{R}} \) on \( M \subset \mathbb{R}^n \) is called dissipative (respectively volume-preserving or conservative), if the relation \( \text{vol}(\varphi^t(A)) < \text{vol}(A) \) (respective \( \text{vol}(\varphi^t(A)) = \text{vol}(A) \)) holds for every set \( A \subset M \) with a positive \( n \)-dimensional volume \( \text{vol}(A) \) and every \( t > 0 \) \((t \in \mathbb{R})\).

A: Let \( \varphi \) in (17.3) be a \( C^r \)-diffeomorphism (i.e.: \( \varphi: M \to M \) is invertible, \( M \subset \mathbb{R}^n \) open, \( \varphi \) and \( \varphi^{-1} \) are \( C^r \)-smooth mappings) and let \( D\varphi(x) \) be the Jacobi matrix of \( \varphi \) in \( x \in M \). The discrete system (17.3) is dissipative if \(|\det D\varphi(x)| < 1 \) for all \( x \in M \), and conservative if \(|\det D\varphi(x)| \equiv 1 \) in \( M \).