Chapter X. Additive Functionals of Brownian Motion

§1. General Definitions

Although we want as usual to focus on the case of linear BM, we shall for a while consider a general Markov process for which we use the notation and results of Chap. III.

(1.1) Definition. An additive functional of $X$ is a $\mathbb{R}_+^g$-valued, $(\mathcal{F})$-adapted process $A = \{A_t, t \geq 0\}$ defined on $\Omega$ and such that

i) it is a.s. non-decreasing, right-continuous, vanishing at zero and such that $A_t = A_{\xi-}$ on $[\xi \leq t]$;

ii) for each pair $(s, t)$, $A_{s+t} = A_t + A_s \diamond \theta_t$ a.s.

A continuous additive functional (abbreviated CAF) is an additive functional such that the map $t \mapsto A_t$ is continuous.

Remark. In ii) the negligible set depends on $s$ and $t$, but by using the right-continuity it can be made to depend only on $t$.

The condition $A_t = A_{\xi-}$ on $[\xi \leq t]$ means that the additive functional does not increase once the process has left the space. Since by convention $\int f(\Delta) = 0$ for any Borel function on $E$, if $\Gamma$ is a Borel subset of $E$, this condition is satisfied by the occupation time of $\Gamma$, namely $A_t = \int_0^t 1_{\Gamma}(X_s) ds$, which is a simple but very important example of a CAF. In particular $A_t = t \wedge \xi$, which corresponds to the special case $\Gamma = E$, is a CAF.

Let $X$ be a Markov process with jumps and for $\varepsilon > 0$ put

$$T_\varepsilon = \inf \{ t > 0 : d(X_t, X_{t-}) > \varepsilon \}. $$

Then $T_\varepsilon$ is an a.s. strictly positive stopping time and if we define inductively a sequence $(T_n)$ by

$$T_1 = T_\varepsilon, \quad T_n = T_{n-1} + T_\varepsilon \circ \theta_{T_{n-1}},$$

the reader will prove that $A_t = \sum_{1}^{\infty} 1_{\{T_n \leq t\}}$ is a purely discontinuous additive functional which counts the jumps of magnitude larger than $\varepsilon$ occurring up to time $t$.

We shall now give the fundamental example of the local time of Brownian motion which was already defined in Chap. VI from the stochastic calculus.
point of view. Actually, all we are going to say is valid more generally for linear Markov processes $X$ which are also continuous semimartingales such that $(X, X)_t = \int_0^t \phi(X_s) \, ds$ for some function $\phi$ and may even be extended further by time-changes (Exercise (1.25) Chap. XI). This is in particular the case for the OU process or for the Bessel processes of dimension $d \geq 1$ or the squares of Bessel processes which we shall study in Chap. XI. The reader may keep track of the fact that the following discussion extends trivially to these cases.

We now consider the BM as a Markov process, that is to say we shall work with the canonical space $W = C(\mathbb{R}_+, \mathbb{R})$ and with the entire family of probability measures $\mathbb{P}_a$, $a \in \mathbb{R}$.

With each $\mathbb{P}_a$, we may, by the discussion in Sect. 1, Chap. VI, associate a process $L$ which is the local time of the martingale $B$ at zero, namely, such that

$$|B_t| = |a| + \int_0^t \text{sgn}(B_s) \, dB_s + L_t \quad \mathbb{P}_a\text{-a.s.}$$

Actually, $L$ may be defined simultaneously for every $\mathbb{P}_a$, since, thanks to Corollary (1.9) in Chap. VI,

$$L_t = \lim_{k \to \infty} \frac{1}{2\epsilon_k} \int_0^t 1_{-\epsilon_k, \epsilon_k}(B_s) \, ds \quad \text{a.s.,}$$

where $\{\epsilon_k\}$ is any sequence of real numbers decreasing to zero. The same discussion applies to the local time at $a$ and yields a process $L^a$. By the results in Chap. VI, the map $(a, t) \rightarrow L_t^a$ is a.s. continuous.

Each of the processes $L^a$ is an additive functional and even a strong additive functional which is the content of

(1.2) Proposition. If $T$ is a stopping time, then, for every $a$

$$L_{T+S}^a = L_T^a + L_S^a(\theta_T) \quad \mathbb{P}_b\text{-a.s.}$$

for every $b$ and every positive random variable $S$.

Proof. Set $I(\epsilon) = \lfloor a - \epsilon, a + \epsilon \rfloor$; it follows from Corollary (1.9) in Chap. VI that if $T$ is a stopping time

$$L_{T+S}^a = L_T^a + \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^S 1_{I(\epsilon)}(B_u(\theta_T)) \, du \quad \mathbb{P}_b\text{-a.s.}$$

for every $b$. By the Strong Markov property of BM

$$P_b \left[ L_S^a(\theta_T) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^S 1_{I(\epsilon)}(B_u(\theta_T)) \, du \right] = 1.$$

Consequently,

$$L_{T+S}^a = L_T^a + L_S^a \circ \theta_T \quad \text{a.s.}$$