In this chapter we prove the fundamental necessary condition of optimality for optimal control problems — Pontryagin Maximum Principle (PMP). In order to obtain a coordinate-free formulation of PMP on manifolds, we apply the technique of Symplectic Geometry developed in the previous chapter. The first classical version of PMP was obtained for optimal control problems in \( \mathbb{R}^n \) by L. S. Pontryagin and his collaborators [15].

12.1 Geometric Statement of PMP and Discussion

Consider the optimal control problem stated in Sect. 10.1 for a control system

\[
\dot{q} = f_u(q), \quad q \in M, \quad u \in U \subset \mathbb{R}^m, \tag{12.1}
\]

with the initial condition

\[
q(0) = q_0. \tag{12.2}
\]

Define the following family of Hamiltonians:

\[
h_u(\lambda) = \langle \lambda, f_u(q) \rangle, \quad \lambda \in T^*_q M, \quad q \in M, \quad u \in U.
\]

In terms of the previous section,

\[
h_u(\lambda) = f_u^*(\lambda).
\]

Fix an arbitrary instant \( t_1 > 0 \).

In Sect. 10.2 we reduced the optimal control problem to the study of boundary of attainable sets. Now we give a necessary optimality condition in this geometric setting.

**Theorem 12.1 (PMP).** Let \( \bar{u}(t), t \in [0, t_1], \) be an admissible control and \( \bar{q}(t) = q_{\bar{u}}(t) \) the corresponding solution of Cauchy problem (12.1), (12.2). If
If \( u(t) \) is an admissible control and \( \lambda_t \) a Lipschitzian curve in \( T^*M \) such that conditions (12.3)–(12.5) hold, then the pair \((u(t), \lambda_t)\) is said to satisfy PMP. In this case the curve \( \lambda_t \) is called an extremal, and its projection \( \tilde{q}(t) = \pi(\lambda_t) \) is called an extremal trajectory.

Remark 12.2. If a pair \((\tilde{u}(t), \lambda_t)\) satisfies PMP, then

\[
\varphi(\lambda_t) = \max_{u \in \tilde{U}} h_u(\lambda_t), \quad t \in [0, t_1].
\]

Indeed, since the admissible control \( \tilde{u}(t) \) is bounded, we can take maximum in (12.5) over the compact \( \{\tilde{u}(t) \mid t \in [0, t_1]\} = \tilde{U} \). Further, the function

\[
\varphi(\lambda) = \max_{u \in \tilde{U}} h_u(\lambda)
\]

is Lipschitzian w.r.t. \( \lambda \in T^*M \). We show that this function has zero derivative. For any admissible control \( u(t) \),

\[
\varphi(\lambda_t) \geq h_u(\tau)(\lambda_t), \quad \varphi(\lambda_\tau) = h_u(\tau)(\lambda_\tau),
\]

thus

\[
\frac{\varphi(\lambda_t) - \varphi(\lambda_\tau)}{t - \tau} \geq \frac{h_u(\tau)(\lambda_t) - h_u(\tau)(\lambda_\tau)}{t - \tau}, \quad t > \tau.
\]

Consequently,

\[
\left. \frac{d}{dt} \right|_{t=\tau} \varphi(\lambda_t) \geq \{h_u(\tau), h_u(\tau)\} = 0
\]

if \( \tau \) is a differentiability point of \( \varphi(\lambda_t) \). Similarly,

\[
\frac{\varphi(\lambda_t) - \varphi(\lambda_\tau)}{t - \tau} \leq \frac{h_u(\tau)(\lambda_t) - h_u(\tau)(\lambda_\tau)}{t - \tau}, \quad t < \tau,
\]

thus

\[
\left. \frac{d}{dt} \right|_{t=\tau} \varphi(\lambda_t) \leq 0.
\]

So

\[
\left. \frac{d}{dt} \right|_{t=\tau} \varphi(\lambda_t) = 0,
\]

and identity (12.6) follows.