Examples of Optimal Control Problems on Compact Lie Groups

19.1 Riemannian Problem

Let $M$ be a compact Lie group. The invariant scalar product $\langle \cdot, \cdot \rangle$ in the Lie algebra $\mathcal{M} = T_{\text{Id}}M$ defines a left-invariant Riemannian structure on $M$:

$$\langle qu, qv \rangle_q \overset{\text{def}}{=} \langle u, v \rangle, \quad u, v \in \mathcal{M}, \quad q \in M, \quad qu, qv \in T_qM.$$  

So in every tangent space $T_qM$ there is a scalar product $\langle \cdot, \cdot \rangle_q$. For any Lipschitzian curve

$q : [0, 1] \rightarrow M$

its Riemannian length is defined as integral of velocity:

$$l = \int_0^1 |\dot{q}(t)| \, dt,$$

$$|\dot{q}| = \sqrt{\langle \dot{q}, \dot{q} \rangle}.$$  

The problem is stated as follows: given any pair of points $q_0, q_1 \in M$, find the shortest curve in $M$ that connects $q_0$ and $q_1$.

The corresponding optimal control problem is as follows:

$$\dot{q} = qu, \quad q \in M, \quad u \in \mathcal{M}, \quad q(0) = q_0, \quad q(1) = q_1, \quad q_0, q_1 \in M \text{ fixed},$$

$$l(u) = \int_0^1 |u(t)| \, dt \rightarrow \text{min}.$$  

First of all, we prove existence of optimal controls. Parametrizing trajectories of control system (19.1) by arc length, we see that the problem with unbounded admissible control $u \in \mathcal{M}$ on the fixed segment $t \in [0, 1]$ is equivalent to the problem with the compact space of control parameters $\mathcal{U} = \{|u| = 1\}$ and free terminal time. Obviously, afterwards we can extend the set of control
parameters to \( U = \{ \| u \| \leq 1 \} \) so that the set of admissible velocities \( f_{U}(q) \) become convex. Then Filippov's theorem implies existence of optimal controls in the problem obtained, thus in the initial one as well.

By Cauchy-Schwartz inequality,

\[
(l(u))^2 = \left( \int_{0}^{1} |u(t)| \, dt \right)^2 \leq \int_{0}^{1} |u(t)|^2 \, dt,
\]

moreover, the equality occurs only if \( |u(t)| = \text{const} \). Consequently, the Riemannian problem \( l \rightarrow \min \) is equivalent to the problem

\[
J(u) = \frac{1}{2} \int_{0}^{1} |u(t)|^2 \, dt \rightarrow \min. \tag{19.4}
\]

The functional \( J \) is more convenient than \( l \) since \( J \) is smooth and its extremals are automatically curves with constant velocity. In the sequel we consider the problem with the functional \( J \): (19.1)–(19.4). The Hamiltonian of PMP for this problem has the form:

\[
h_{u}(a, q) = (\dot{a}, qu) + \frac{\nu}{2} |u|^2 = (a, u) + \frac{\nu}{2} |u|^2.
\]

The maximality condition of PMP is:

\[
h^{\nu}_{u(t)}(a(t), q(t)) = \max_{v \in \mathcal{M}} ((a(t), v) + \frac{\nu}{2} |v|^2), \quad \nu \leq 0.
\]

(1) Abnormal case: \( \nu = 0 \).

The maximality condition implies that \( a(t) \equiv 0 \). This contradicts PMP since the pair \( (\nu, a) \) should be nonzero. So there are no abnormal extremals.

(2) Normal case: \( \nu = -1 \).

The maximality condition gives \( u(t) \equiv a(t) \), thus the maximized Hamiltonian is smooth:

\[
H(a) = \frac{1}{2} |a|^2.
\]

Notice that the Hamiltonian \( H \) is invariant (does not depend on \( q \)), which is a corollary of left-invariance of the problem.

Optimal trajectories are projections of solutions of the Hamiltonian system corresponding to \( H \). This Hamiltonian system has the form (see (18.18)):

\[
\begin{cases}
\dot{q} = qa, \\
\dot{a} = [a, a] = 0.
\end{cases}
\]

Thus optimal trajectories are left translations of one-parameter subgroups in \( M \):

\[
q(t) = q_{0} e^{ta}, \quad a \in \mathcal{M},
\]

recall that an optimal solution exists. In particular, for the case \( q_{0} = \text{Id} \), we obtain that any point \( q_{1} \in M \) can be represented in the form