12 Orthogonal Series and Special Functions

12.1 Orthogonal Systems

Assumptions. \( \varphi_k(x) \), \( w(x) \) are real and continuous, \( f(x) \) real and piece-wise continuous.

Definition.

The set \( \{ \varphi_k(x) \} \) is an orthogonal system on \((a, b)\) [finite or infinite interval] with respect to the weight function \( w(x) \geq 0 \) if

\[
(i) \quad (\varphi_k, \varphi_n)_w = \int_a^b \varphi_k(x) \varphi_n(x) w(x) dx = 0, \quad n \neq k, \quad (\varphi_k, \varphi_n \text{ w-orthogonal})
\]

\[
(ii) \quad N_k = \| \varphi_k \|_w^2 = \int_a^b \varphi_k^2(x) w(x) dx > 0
\]

General Fourier Series (complete orthogonal system)

\[
(12.1) \quad f(x) = \sum_{k=1}^{\infty} c_k \varphi_k(x), \quad c_k = \frac{(f, \varphi_k)_w}{\| \varphi_k \|_w^2} = \frac{1}{\| \varphi_k \|_w^2} \int_a^b f(x) \varphi_k(x) w(x) dx
\]

Here \( c_k \) are the (general) Fourier coefficients of \( f(x) \).

Approximation in mean

If \( s_n(x) = a_1 \varphi_1(x) + \ldots + a_n \varphi_n(x) \), then

1. \( Q_n = \| f - s_n \|_w^2 = \int_a^b [f(x) - s_n(x)]^2 w(x) dx \) is minimal \( \Leftrightarrow a_k = c_k, \) \( k = 1, \ldots, n \)

2. \( \min Q_n = \int_a^b f^2(x) w(x) dx - \sum_{k=1}^n N_k c_k^2 \)

The system \( \{ \varphi_k(x) \} \) is complete if \( \lim_{n \to \infty} \int_a^b [f(x) - s_n(x)]^2 w(x) dx = 0 \)

(with \( a_k = c_k \)), i.e. \( s_n \to f \) in mean as \( n \to \infty \).
Theorem.

3. \( \{ \varphi_k(x) \}_{k=1}^{\infty} \) orthogonal system \( \Rightarrow \sum_{k=1}^{\infty} N_k c_k^2 \leq \int_{a}^{b} f^2(x)w(x)dx \) (Bessel's inequality)

4. \( \{ \varphi_k(x) \}_{k=1}^{\infty} \) complete orthogonal system \( \Rightarrow \)

\[
(i) \sum_{k=1}^{\infty} N_k c_k^2 = \int_{a}^{b} f^2(x)w(x)dx \\
(ii) \sum_{k=1}^{\infty} N_k c_k d_k = \int_{a}^{b} f(x)g(x)w(x)dx \] (Parseval's identity)

Orthogonal systems in Linear Spaces

Let \((u, v)\) denote a scalar product of elements \(u\) and \(v\) in a linear space \(H\) and let \(\|u\| = \sqrt{(u, u)}\) denote the norm of \(u\).

1. Let \(\{ \varphi_k \}_{k=1}^{n}\) be an orthogonal set (system) in \(H\), i.e. \((\varphi_i, \varphi_j) = 0\) for \(i \neq j\).

2. Let \(V = \left\{ \sum_{k=1}^{n} a_k \varphi_k : a_k \text{ constants} \right\}\) be the linear subspace spanned by \(\{ \varphi_k \}_{k=1}^{n}\)

3. \(Pu = \sum_{k=1}^{n} \frac{(u, \varphi_k)}{\|\varphi_k\|^2} \varphi_k\) (the orthogonal projection of \(u\) onto \(V\))

Projection theorem

4. \(Pu \in V \quad 5. (u - Pu) \perp V\) i.e. \((u - Pu, v) = (u - Pu, \varphi_k) = 0\), all \(v \in V, k = 1, 2, \ldots, n\)

6. \(\min_{v \in V} \|u - v\|^2 = \|u - Pu\|^2 = \|u\|^2 - \sum_{k=1}^{n} \frac{(u, \varphi_k)^2}{\|\varphi_k\|^2}\)

7. \(\sum_{k=1}^{n} \frac{(u, \varphi_k)^2}{\|\varphi_k\|^2} \leq \|u\|^2\) (Bessel's inequality)

8. \(\{ \varphi_k \}_{k=1}^{\infty}\) complete: \(\sum_{k=1}^{\infty} \frac{(u, \varphi_k)^2}{\|\varphi_k\|^2} = \|u\|^2\) (Parseval's identity)

The Sturm-Liouville eigenvalue problem

Assumptions.

(i) \(p(x), p'(x), q(x), w(x)\) real, continuous in \([a, b]\)

(ii) \(p(x) > 0, q(x) \geq 0\) in \([a, b]\), \(w(x) > 0\) in \((a, b)\)

(iii) \(A, B, C, D\) real non-negative constants, \(A^2 + B^2 > 0, C^2 + D^2 > 0\)