4 Fundamentals of Multidimensional CFD-Codes

4.1 Conservation Equations

The abbreviation CFD stands for computational fluid dynamics which indicates the numerical solution of multidimensional flow problems that may be of unsteady and turbulent nature. In general, multidimensional flow problems are governed by conservation principles for mass energy and momentum. The application of these principles results in a set of partial differential equations in terms of time and space that need to be integrated numerically as they are too complex to be solved analytically.

For the sake of simplicity and clearness the subsequent analysis is based on single-component, single-phase flows. However, it should be noted that in engine combustion chambers the gas phase usually consists of multiple components that are subject to chemical reactions. Furthermore, in direct injection engines two-phase flows with evaporating fuel droplets are encountered. In order to take account of these effects, additional transport- or source-terms have to be added to the conservation equations as it is indicated in Sect. 4.5.

Mass Conservation

The continuity equation based on mass conservation can be derived for an infinitesimal volume element \( dV = dx_1 \, dx_2 \, dx_3 \) as indicated in Fig. 4.1. In a Eulerian approach, i.e. with a coordinate system fixed in space, the control volume is fixed in space as well. It can be passed by the flow without resistance and the mass within the control volume will increase if the inflow exceeds the outflow. In the opposite case it will decrease. Either case is associated with a density change in the control volume. The mass balance reads

\[
\frac{\partial}{\partial t} (dx_1 \, dx_2 \, dx_3 \, \rho) = d\dot{m}_{x_1} + d\dot{m}_{x_2} + d\dot{m}_{x_3},
\]

where

\[
d\dot{m}_{x_i} = \left( \dot{m}_{x_i} \right)_{x_i} - \left( \dot{m}_{x_i} \right)_{x_i + dx_i} = dx_2 \, dx_3 \left( \rho v_1 \right)_{x_i} - dx_2 \, dx_3 \left( \rho v_1 \right)_{x_i + dx_i},
\]

and the second and third terms on the right hand side of Eq. 4.1 are treated accordingly. Furthermore, the second term on the right hand side of Eq. 4.2 can be expressed by a Taylor series.
\[ (\rho v_i)_{x_i + dx_i} = (\rho v_i)_{x_i} + \frac{\partial}{\partial x_i} (\rho v_i)_{x_i} \, dx_i, \quad (4.3) \]

and we obtain
\[ \dot{m}_{v_i} = -dx_2 \, dx_3 \frac{\partial}{\partial x_i} (\rho v_i)_{x_i} \, dx_i. \quad (4.4) \]

The terms \( \dot{m}_{v_i} \) and \( \dot{m}_{v_i} \) are again treated accordingly. Since the size of the control volume is constant with time, the \( dx_i \) in Eq. 4.1 can be taken out of the differential operator. Combining Eqs. 4.1 and 4.4 then yields the general form of mass conservation:
\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho v_i) = 0. \quad (4.5) \]

In Eq. 4.5 and throughout this chapter the Einstein convention will be utilized. It states that whenever the same index appears twice in any term, summation over the range of that index is implied, i.e.
\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho v_i) = \frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_1)}{\partial x_1} + \frac{\partial (\rho v_2)}{\partial x_2} + \frac{\partial (\rho v_3)}{\partial x_3} = 0. \quad (4.6) \]

In many applications the fluid may be treated as incompressible. This is true not only for flows of liquids, whose compressibility may indeed be neglected, but also for gases if the Mach number is below approx. 0.3 [3]. Applying the chain rule, Eq. 4.5 can be written as
\[ \frac{\partial \rho}{\partial t} + v_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial v_i}{\partial x_i} = 0, \quad (4.7) \]

and for incompressible flows \((\rho = \text{const.})\) it reduces to
\[ \text{div} (\vec{v}) \equiv \frac{\partial v_i}{\partial x_i} = 0. \quad (4.8) \]

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**Fig. 4.1.** Mass fluxes entering and exiting the control volume