Since their introduction by Engle and Bollerslev models with autoregressive, conditional heteroscedasticity (\textit{autoregressive conditional heteroscedasticity models} or ARCH) have been successfully applied to financial market data. Thus it is natural to discuss option pricing models where the underlying instrument follows an ARCH process. From an empirical point of view the form of the \textit{news impact curve}, which is defined as a function of the current volatility dependent on yesterday’s returns, is the dominant factor in determining the price. It is important, for example, to know whether the news impact curve is symmetric or asymmetric. In order to avoid inaccurate pricing due to asymmetries it is necessary to use flexible volatility models. In this way EGARCH models (see Section 12.2) can be used when stock prices and volatility are correlated. This model however has a weakness that the problem of the stationarity conditions and the asymptotic of the Quasi-Maximum-Likelihood-Estimator (QMLE) is not yet completely solved. Another Ansatz, as in the Threshold GARCH-Models, is to introduce thresholds in the news impact curve to create flexible asymmetry.

In this chapter we would like to concentrate on the specification of the volatility. We will concentrate on a TGARCH process and produce extensive Monte Carlo simulations for three typical parameter groups. In particular we will compare the simulated GARCH option prices with option prices based on the simulations from TGARCH and Black-Scholes models. In the empirical section of the chapter we will show that the market price of call options indeed reflect the asymmetries that were discovered in the news impact curve of the DAX time series.
14.1 Valuing Options with ARCH-Models

Consider an economy in discrete time in which interest and proceeds are paid out at the end of every constant, equally long time interval. Let \( S_t, t = 0, 1, 2, \ldots \) be the price of the stock at time \( t \) and \( Y_t = (S_t - S_{t-1})/S_{t-1} \) the corresponding one period return without dividends. Assume that a price for risk exists in the form of a risk premium which is added to the risk free interest rate \( r \) to obtain the expected return of the next period. It seems reasonable to model the risk premium dependent on the conditional variance. As a basis we assume an ARCH-M-Model (see Section 12.2.3) with a risk premium, which is a linear function of the conditional standard deviation:

\[
Y_t = r + \lambda \sigma_t + \varepsilon_t \quad (14.1)
\]

\[
\mathcal{L}(\varepsilon_t | \mathcal{F}_{t-1}) = N(0, \sigma^2_t) \quad (14.2)
\]

\[
\sigma^2_t = \omega + \alpha \varepsilon^2_{t-1} + \beta \sigma^2_{t-1} \quad (14.3)
\]

In (14.3) \( \omega, \alpha \) and \( \beta \) are constant parameters that satisfy the stationarity and non-negativity conditions. The constant parameter \( \lambda \) can be understood as the price of one unit of risk. \( \mathcal{F}_t \) indicates, as usual, the set of information available up to and including time \( t \). In order to simplify the notation our discuss will be limited to the GARCH(1,1) case.

The above model is estimated under the empirical measure \( P \). In order to deal with a valuation under no arbitrage, similar to Black-Scholes in continuous time (see Section 6.1), assumptions on the valuation of risk must be met. Many studies have researched option pricing with stochastic volatility under the assumption that the volatility has a systematic risk of zero, that is, the risk premium for volatility is zero. Duan (1995) has identified an equivalent martingale measure \( Q \) for \( P \) under the assumption that the conditional distribution of the returns are normal and in addition it holds that

\[
\text{Var}^P(Y_t | \mathcal{F}_{t-1}) = \text{Var}^Q(Y_t | \mathcal{F}_{t-1}) \quad (14.4)
\]

\( P \) a.s.. He shows that under this assumption a representative agent with, for example, constant relative risk aversion and a normally distributed relative change of aggregate consumption maximizes his expected utility. The assumption (14.4) contains a constant risk premium for the volatility that directly enters its mean.

In order to obtain a martingale under the new measure a new error term, \( \eta_t \), needs to be introduced that captures the effect of the time varying risk premium. When we define \( \eta_t = \varepsilon_t + \lambda \sigma_t \), (14.4) leads to the following model