Constructions of Knotted Surfaces

In this chapter, we discuss many of the known methods for constructing knotted surfaces in 4-space. These constructions include spinning and its generalizations, ribbon and simply knotted surfaces, and the connected sum. The chapter also contains a section on Seifert solids (hypersurfaces), and other related constructive aspects.

In knot theory, many constructions are invented to supply examples with desired values or properties of various invariants. This makes it difficult to catalog constructions separately from discussions on invariants (Chapters 3 and 4). Some constructions, however, appear repeatedly in many aspects, and have become centers of study themselves. Some others are direct analogues from classical knot theory and it seems convenient to be mentioned separately here. These situations motivate this chapter.

2.1 Spinning Constructions

Spinning constructions have played key roles in many aspects in the study of knotted surfaces. In this section, we describe the spinning constructions in historical order.

2.1.1 Artin's Original Spinning

Let k be an arc embedded in the 3-dimensional half-space \( \mathbb{R}_+^3 = \{(x_1, x_2, x_3, x_4) | x_3 \geq 0, x_4 = 0\} \). When half-space is rotated around the plane \( \mathbb{R}^2 = \{(x_1, x_2, x_3, x_4) | x_3 = 0, x_4 = 0\} \) in \( \mathbb{R}^4 \), the continuous trace of k forms a locally flat 2-sphere. This 2-sphere, \( S(k) \), is said to be derived from k by spinning. An example is illustrated in Fig. 2.1. E. Artin [Ar25] used this method to construct a 2-knot whose knot group is isomorphic to a given classical knot group; the knot group \( \pi_1(\mathbb{R}^4 \setminus S(k)) \) of \( S(k) \) is isomorphic to \( \pi_1(\mathbb{R}_+^3 \setminus k) \) which, in turn, is isomorphic to the knot group of a knot associated with the knotted arc k. (A knot is associated with a knotted arc k if it is
obtained by connecting the endpoints of \( k \) in an obvious way with an arc in \( \mathbb{R}^2 \). In Fig. 2.1, the associated knot is a trefoil.)

![Fig. 2.1. A knotted arc](image)

### 2.1.2 Twist-spinning

Artin's spinning construction was generalized by Zeeman [Ze65], who introduced twisting while the knotted arc \( k \) is spun. Put the knotted part of \( k \) in a 3-ball as in Fig. 2.2 (the figure is an example for a trefoil knot) and twist it \( m \) times as it spins. Another 2-knot in \( \mathbb{R}^4 \) is obtained which is said to be derived from \( k \) by \( m \)-twist spinning.

We denote by \( S_m(k) \) the \( m \)-twist spun \( k \). The knot group of \( S_m(k) \) is not, in general, isomorphic to the group \( \pi_1(\mathbb{R}_+^3 \setminus k) \) but is a quotient group of it:

\[
\pi_1(\mathbb{R}^4 \setminus S_m(k)) = \pi_1(\mathbb{R}_+^3 \setminus k) / \langle a_0^{-m}a_0^m = a \mid a \in \pi_1(\mathbb{R}_+^3 \setminus k) \rangle
\]

where \( a_0 \) is a meridional element. Since the group \( \pi_1(\mathbb{R}_+^3 \setminus k) \) is generated by a finite number of meridian elements, say \( a_0, a_1, \ldots, a_s \), we may write

\[
\pi_1(\mathbb{R}^4 \setminus S_m(k)) = \pi_1(\mathbb{R}_+^3 \setminus k) / \langle a_0^{-m}a_0^m = a_i \mid i \in \{1, \ldots, s\} \rangle.
\]

For example, if \( k \) is associated with a trefoil knot as in Fig. 2.2, then

\[
\pi_1(\mathbb{R}^4 \setminus S_m(k)) = \langle a, b \mid aba = bab, a^{-m}ba^m = b \rangle.
\]

By definition, if \( m = 0 \), then \( S_0(k) \) is Artin's 2-knot \( S(k) \). If \( m = 1 \), then it is obvious that the knot group \( \pi_1(\mathbb{R}^4 \setminus S_1(k)) \) is abelian. Thus it is natural to conjecture that \( S_1(k) \) is unknotted, and it is indeed the case [Ze65]. In fact, Zeeman proved the following theorem.

**Theorem 2.1. [Ze65]** For \( m \geq 1 \), \( S_m(k) \) is a fibered 2-knot whose closed fiber is the \( m \)-fold cyclic branched covering of \( S^3 \) branched along a knot associated with \( k \).