Extensions of the Classical Theory

1 Introduction

This chapter deals with four different topics.

Our first subject is the algebraic approach to Wiener's theory (Sections 2–9). After some preliminaries on Banach algebras we present an algebraic form of Wiener's Approximation Theorem 11.8.3. The subsequent treatment in the context of Banach algebras makes it natural to include Beuding's extension [1938] of the theorem to weighted $L^1$ spaces. Our discussion includes the necessary parts of Gelfand's theory [1939], [1941a] of maximal ideals, complemented by some of Shilov's results [1940], [1947] on so-called minimal ideals.

Sections 10–13 treat an extension of Wiener's theory to the case of rapidly decreasing kernels. It is due to Pitt [1938a], [1958] and plays an important role in Chapter VI. Let $K \in L^1(\mathbb{R})$ be a Wiener kernel, so that $K(t) \neq 0$ for all real $t$, and let $S$ be such that the convolution

$$K * S(x) = \int_{\mathbb{R}} K(x - y)S(y)dy$$

exists and tends to $A \int_{\mathbb{R}} K(y)dy$ as $x \to \infty$. \hfill (1.1)

In the Wiener–Pitt Theorem 11.8.4 one obtains convergence of $S(x)$ to $A$ from (1.1) under the condition that $S$ be bounded and satisfy an appropriate Tauberian condition. In the case of rapidly decreasing kernels such as $K(x) = e^{-x^2}$, one need not postulate that $S$ is bounded: the boundedness can be derived from a condition of at most exponential growth. At the same time, the function $K(x - y)$ may be replaced by a function $J(x, y)$ of more general type, but such a function must be well-approximated by a suitable 'difference kernel'. Important examples are provided by the kernels for Borel summability and other circle methods; see Section VI.16.

In Sections 14–21 we discuss a functional-analytic method to reduce the general case of a Tauberian theorem to the simpler case of bounded functions, or in this case, sequences. The method goes back to the Polish school of functional analysis;
for its implementation, an appropriate theory of so-called \( FK\)-spaces was developed by Wilansky, Zeller and others. Cf. the books by Zeller and Beekmann [1958/70], Wilansky [1984], and Boos [2000]. In the hands of Meyer-König and Zeller, the technique turned out to be effective in the treatment of Tauberian theorems for lacunary series.

The final Sections 22–26 are devoted to some striking Tauberians of different character. The first theorem, due to Erdős, Feller and Pollard [1949], was inspired by renewal theory; cf. Feller [1950/68] (chapter 13). The second result is an unusual Tauberian theorem due to Milin [1970], [1971]. It is important in the theory of univalent functions; cf. Duren [1983].

2 Preliminaries on Banach Algebras

For an algebraic formulation of Wiener’s theorem we need some simple notions from the theory of (complex) Banach algebras. The basics can be found in many books, for example Rudin [1966/87] or [1973/91]. Additional references will be given in Section 5.

A \textsc{Banach Algebra} \( A \) is a complex Banach space in which any two elements can be multiplied. The multiplication must satisfy the usual associative and distributive laws, and it is customary to require that the norm of the product \( xy \) satisfy the inequality

\[
\|xy\| \leq \|x\|\|y\|.
\]

We only consider \textsc{commutative} Banach algebras: \( xy = yx \) for all \( x, y \in A \). A Banach algebra may or may not have a \textsc{unit element}, that is, an element \( e \) such that \( ex = xe = x \) for all \( x \in A \). If there is a unit \( e \), we will require that \( \|e\| = 1 \).

\textbf{Examples 2.1.} The ‘Wiener Algebra’ \( A_W \) consists of the continuous functions \( f \) of period \( 2\pi \) with absolutely convergent Fourier series,

\[
f(t) = \sum_{-\infty}^{\infty} a_n e^{int}, \quad \text{with norm } \|f\| = \sum_{-\infty}^{\infty} |a_n| < \infty.
\]

The product \( fg \) is defined in the ordinary way. Hence if \( g \) has Fourier coefficients \( b_n \), then

\[
f(t)g(t) = \sum_{m} a_m e^{int} \sum_{k} b_k e^{ikt} = \sum_{n} \left( \sum_{k} (a_{n-k}b_k) \right) e^{int},
\]

\[
\|fg\| = \sum_{n} \left| \sum_{k} a_{n-k}b_k \right| \leq \sum_{k} \left( \sum_{n} |a_{n-k}| \right) |b_k| = \|f\|\|g\|.
\]

The function \( e = 1 \) serves as unit.

A more important example is the \textsc{convolution algebra} \( L^1(\mathbb{R}) \), that is, the normed space \( L^1(\mathbb{R}) \), furnished with the product given by convolution: