On the General Theory of Asymmetric Gyros

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1. Introduction

The problem of finding particular solutions of the general equations of gyrodynamics is generally less difficult if one looks for motions with given characteristic properties. In doing so one has the possibility to find solutions which have an evident kinematic and geometrical meaning, and this is interesting since it is possible to understand the significance of the corresponding motions.

One of the most interesting problems in dynamics of rigid bodies is the determination of the motions of precession, i.e. of the motions which involve the rotation about an axis which is itself rotating about an axis fixed in space. This problem includes, as a particular case, uniform and non-uniform rotations which will not be dealt with in the following.

I have studied the general problem of the motions of precession for an unsymmetrical rigid body of any shape, subjected to a certain set of external active forces. This case includes, in particular, the problem of the heavy top and that of the Coriolis and Lorentz forces.

The general problem is mathematically very difficult. After several attempts I became convinced that the least complicated approach is a method that at first sight seems to be less plain than others.

2. General Theory

I think that it is possible to achieve one’s purpose by expressing the basic kinematic condition characterizing the motions of precession only by means of the components \( p, q, r \) of the angular velocity of the body with respect to a lefthanded coordinate system fixed in the body, although this approach seems at first sight less natural than that of postulating that the angle between a straight line fixed in the body and a line fixed in space be independent of time, which is the usual
condition characterizing precession motions. It is possible to show that this condition is [1]
\[ \frac{p \dot{q} - g \dot{p} + r (p^2 + q^2)}{(p^2 + q^2) \sqrt{p^2 + q^2}} = \cot \theta, \]
(1)
where $\theta$ is a non-zero constant.

From (1) it follows that the necessary and sufficient condition for a rigid motion to be a precession is that every unit vector $u$, independent of the time, can be represented in the form

\[ u = \sin \theta \left[ (f_1 p + f_2 q) \mathbf{i} + (f_1 q - f_2 p) \mathbf{j} \right] + f_3 \mathbf{k}, \]
(2)
where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors of the reference system fixed in the body, $\theta$ the constant introduced in (1) and $\theta$ denotes the angle between $u$ and the precession axis.

Let $\omega$ be the angular velocity of the body, $V$ an invariant space vector and $\sigma$ the matrix of inertia with respect to the fixed point, whose components are the moments of inertia $A, B, C$ and the products of inertia, $A', B', C'$. I assume that the ellipsoid of inertia is not an ellipsoid of revolution\(^1\). I will suppose that the basic vectorial equation of motion, expressing the theorem of angular momentum, is

\[ \frac{d\sigma \omega}{dt} = M_0(\omega, V), \]
(5)
where $M_0$ is a vectorial function of $\omega$ and $V$, depending on the point $O$.

Further I will assume that the function $M_0$ is such that two first integrals exist: the integral of energy

\[ \sigma \omega \cdot \omega = N(V), \]
(6)
and another first integral of the form

\[ \sigma \omega \cdot V = L(V). \]
(7)

In Eqs. (6) and (7), $N(V), L(V)$ are scalar functions of the vector $V$.

In order to determine the precessions it is convenient to express the vector $V$ in the form (2). Then the right hand members of Eqs. (5),

\(^1\) In the case that the ellipsoid of inertia is an ellipsoid of revolution, see [2].