Chapter II.3 Angles, Limits, Cones and Joins

In this chapter we examine how upper curvature bounds influence the behaviour of angles, limits of sequences of spaces, and the cone and join constructions described in (I.5). We then use the results concerning limits and cones to describe the space of directions at a point in a CAT(κ) space.

Angles in CAT(κ) Spaces

In CAT(κ) spaces, angles exist in the following strong sense.

3.1 Proposition. Let $X$ be a CAT(κ) space and let $c : [0, a] \to X$ and $c' : [0, a'] \to X$ be two geodesic paths issuing from the same point $c(0) = c'(0)$. Then the $\kappa$-comparison angle $\angle_{c(0)}^{(\kappa)}(c(t), c'(t))$ is a non-decreasing function of both $t, t' \geq 0$, and the Alexandrov angle $\angle(c, c')$ is equal to $\lim_{t,t' \to 0} \angle_{c(0)}^{(\kappa)}(c(t), c'(t)) = \lim_{t \to 0} \angle_{c(0)}^{(\kappa)}(c(t), c'(t))$. Hence, in the light of (1.2.9),

$$\angle(c, c') = \lim_{t \to 0} 2 \arcsin \frac{1}{2t} d(c(t), c'(t)).$$

Proof. Immediate from 1.7(3) and the fact that one can take comparison triangles in $\mathbb{M}^2$ instead of $\mathbb{E}^2$ in the definition (I.1.12) of the Alexandrov angle (see I.2.9).

3.2 Notation for Angles. For the convenience of the reader we recall the following notation. Let $p, x, y$ be points of a metric space $X$ such that $p \neq x, p \neq y$.

- $\angle_p^{(\kappa)}(x, y)$ denotes the comparison angle in $\mathbb{M}^2$ (see I.2.15);
- $\angle_p(x, y) = \angle_0^{(\kappa)}(x, y)$ denotes the comparison angle in $\mathbb{E}^2$ (see I.1.12);
- if there are unique geodesic segments $[p, x]$ and $[p, y]$, then we write $\angle_p(x, y)$ to denote the (Alexandrov) angle between these segments (see I.1.12).

Recall that a real-valued function $f$ on a topological space $Y$ is said to be upper semicontinuous if $f(y) \geq \lim \sup_{n \to \infty} f(y_n)$ whenever $y_n \to y$ in $Y$.

3.3 Proposition. Let $X$ be a CAT(κ) space. For all points $p, x, y \in X$ with $\max\{d(p, x), d(p, y)\} < D_\kappa$, 

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(1) **the function** \( (p, x, y) \mapsto \angle_p(x, y) \) **is upper semicontinuous, and**

(2) **for fixed** \( p \in X \), **the function** \( (x, y) \mapsto \angle_p(x, y) \) **is continuous.**

**Proof.** (1) **Let** \((x_n), (y_n)\) **and** \((p_n)\) **be sequences of points converging to** \(x, y\) **and** \(p\) **respectively. Let** \(c, c', c_n\) **and** \(c_n'\) **be linear parameterizations** \([0, 1] \to X\) **of the geodesic segments** \([p, x], [p, y], [p_n, x_n]\) **and** \([p_n, y_n]\) **respectively. For** \(t \in (0, 1]\), **let** \(\alpha(t) = \angle_p^{(c)}(c(t), c'(t))\) **and let** \(\alpha_n(t) = \angle_{p_n}^{(c_n)}(c_n(t), c_n'(t))\). According to (3.1), \(\alpha(t)\) **and** \(\alpha_n(t)\) **are non-decreasing functions of** \(t\) **and** \(\alpha := \angle_p(x, y) = \lim_{t \to 0^+} \alpha(t)\) **and** \(\alpha_n := \angle_{p_n}(x_n, y_n) = \lim_{t \to 0} \alpha_n(t)\). **And for fixed** \(t\) **we have** \(\alpha_n(t) \to \alpha(t)\) **as** \(n \to \infty\). **Given** \(\varepsilon > 0\), **let** \(T > 0\) **be such that** \(\alpha(t) - \varepsilon / 2 \leq \alpha\) **for all** \(t \in (0, T]\). **Then for** \(n\) **big enough,** \(\alpha_n(T) \leq \alpha(T) + \varepsilon / 2\), **therefore** \(\alpha_n \leq \alpha_n(T) \leq \alpha(T) + \varepsilon / 2 \leq \alpha + \varepsilon\). **Thus** \(\limsup \alpha_n \leq \alpha\) **as required.**

(2) **We keep the above notations, but we assume that** \(p_n = p\) **for all** \(n\). **Let** \(\beta_n = \angle_p(x, x_n)\) **and** \(\gamma_n = \angle_p(y, y_n)\). **By** (1.7(4)), \(\beta_n \to 0\) **and** \(\gamma_n \to 0\) **as** \(n \to \infty\). **By the triangle inequality for Alexandrov angles,** \(|\alpha - \alpha_n| \leq \beta_n + \gamma_n\). **Hence** \(\lim_{n \to \infty} \alpha_n = \alpha\).

**3.4 Remark.** With regard to part (1) of the preceding proposition, we note that in general it will not be true that \(p \mapsto \angle_p(x, y)\) **is a continuous function. For example, consider the CAT(0) space** \(X\) **obtained by endowing the subset** \(\{(x_1, 0) \mid x_1 \in \mathbb{R}\} \cup \{(x_1, x_2) \mid x_1 \geq 0, x_2 \in \mathbb{R}\}\) **of the plane with the induced length metric from** \(\mathbb{E}^2\). **Let** \(p = (0, 0)\) **and** \(p_n = (-1 / n, 0)\). **Given any** \(x = (x_1, x_2)\) **and** \(y = (y_1, y_2) \in X\) **such that** \(x_1 \geq 0\) **and** \(y_1 \geq 0\), **the angle which** \(x\) **and** \(y\) **subtend at** \(p\) **is equal to the usual Euclidean angle, but** \(\angle_{p_n}(x, y) = 0\) **for all** \(n\).

**The following addendum to (3.3) will be useful later.**

**3.5 Proposition.** **Let** \(X\) **be a CAT(\(\kappa\)) space, let** \(c : [0, \varepsilon] \to X\) **be a geodesic segment issuing from** \(p = c(0)\) **and let** \(y\) **be a point of** \(X\) **distinct from** \(p\) **(with** \(\varepsilon\) **and** \(d(p, y)\) **less than** \(D_k\) **if** \(\kappa > 0\)). **Then,**

\[
\lim_{s \to 0} \angle_p(c(s), y) = \angle_p(c(\varepsilon), y).
\]

**Proof.** **As** \(s \mapsto \angle_p^{(c)}(c(s), y)\) **is non-decreasing,** \(\gamma := \lim_{s \to 0} \angle_p^{(c)}(c(s), y)\) **exists. By** (1.2.9), **we have** \(\gamma = \lim_{s \to 0} \angle_p(c(s), y)\). **This last expression is, by definition, the strong upper angle between** \([p, y]\) **and** \(c\), **which we showed in (1.1.16) to be equal to the Alexandrov angle.**

**3.6 Corollary (First Variation Formula).** **With the notation of the preceding proposition,**

\[
\lim_{s \to 0} \frac{d(c(0), y) - d(c(s), y)}{s}
\]

**exists and is equal to** \(\cos \angle_p(c(s), y)\).

**Proof.** **This follows from the preceding proposition and the Euclidean law of cosines.**