12. \( \lim^n \) and the extension functors \( \text{Ext}^n \)

In this section we first define and analyze in detail the extension products \( \text{Ext}^n(A, X) \) of two inverse systems of modules. We then show that \( \lim^n X \) coincides with \( \text{Ext}^n(\Delta(A), X) \), where \( \Delta(A) \) is the diagonal inverse system. The advantage of this description of \( \lim^n X \) over the description given in 11 lies in the fact that \( \lim^n X \) can be determined using, the same projective resolution of \( \Delta(A) \), for all \( X \).

12.1 The bifunctors \( \text{Ext}^n \)

For any fixed inverse system \( A \in \text{Mod}^\Lambda \), \( \text{Hom}(A, \cdot) : \text{Mod}^\Lambda \to \text{Mod} \) is a covariant functor. To a system \( X \), it assigns the \( R \)-module \( \text{Hom}(A, X) \) consisting of all morphisms \( h : A \to X \). To a morphism \( f : X \to X'' \), it assigns the homomorphism \( \text{Hom}(A, f) : \text{Hom}(A, X) \to \text{Hom}(A, X'') \), which maps \( h \in \text{Hom}(A, X) \) to \( fh \in \text{Hom}(A, X'') \).

LEMMA 12.1. The functor \( \text{Hom}(A, \cdot) \) is additive and left exact. Moreover, if \( A \) is a projective object of \( \text{Mod}^\Lambda \), then \( \text{Hom}(A, \cdot) \) is an exact functor.

Proof. The additivity of \( \text{Hom}(A, \cdot) \) follows from Lemma 11.6. To prove left exactness, consider an exact sequence of inverse systems

\[
0 \to X' \xrightarrow{f'} X \xrightarrow{f} X''.
\]

We must prove that the induced sequence

\[
0 \to \text{Hom}(A, X') \xrightarrow{f'} \text{Hom}(A, X) \xrightarrow{f} \text{Hom}(A, X'').
\]

is exact; here \( f' = \text{Hom}(A, f') \), \( f = \text{Hom}(A, f) \). By Lemma 11.5, the sequence of modules

\[
0 \to X'_\lambda \xrightarrow{f'_\lambda} X_\lambda \xrightarrow{f_\lambda} X''_\lambda
\]

is exact, for every \( \lambda \in \Lambda \). It is well known that, for any module \( A \), the functor \( \text{Hom}(A, \cdot) : \text{Mod} \to \text{Mod} \) is left exact. Therefore, (3) yields exactness of

\[
0 \to \text{Hom}(A\lambda, X'_\lambda) \to \text{Hom}(A\lambda, X_\lambda) \to \text{Hom}(A\lambda, X''_\lambda).
\]
Now let \( h = (h_\lambda) \in \text{Hom}(A, X) \) belong to the kernel of \( \text{Hom}(A, f) \), i.e., \( fh = 0 \). In order to show that \( h \) belongs to the image of \( \text{Hom}(A, f') \), we must find a morphism \( h' = (h'_\lambda) \in \text{Hom}(A, X') \), such that \( f'h' = h \), i.e., \( f'_\lambda h'_\lambda = h_\lambda \). By assumption, \( f_\lambda h_\lambda = 0 \) and therefore, by exactness of (4), there exist unique homomorphisms \( h'_\lambda: A_\lambda \to X'_\lambda \) such that
\[
f'_\lambda h'_\lambda = h_\lambda.
\] (5)

It remains to verify that the homomorphisms \( h'_\lambda \) form a morphism \( h': A \to X' \), i.e., that they satisfy the equalities
\[
p'_{\lambda\lambda'} h'_\lambda = h'_\lambda a_{\lambda\lambda'}, \quad \lambda \leq \lambda',
\] (6)
where \( p'_{\lambda\lambda'} \) and \( a_{\lambda\lambda'} \) denote the bonding homomorphisms of \( X' \) and \( A \), respectively. Using (5), for \( \lambda' \) and \( \lambda \), as well as the fact that \( f' \) and \( h \) are morphisms, one readily obtains
\[
f'_\lambda p'_{\lambda\lambda'} h'_\lambda = p_{\lambda\lambda'} f'_\lambda h'_\lambda = p_{\lambda\lambda'} h'_\lambda = h_\lambda a_{\lambda\lambda'} = f'_\lambda h'_\lambda a_{\lambda\lambda'}.
\] (7)

Since \( f'_\lambda \) is a monomorphism, (7) implies the desired formula (6). The remaining parts of the proof of the left exactness of \( \text{Hom}(A, .) \) are straightforward and we omit them.

Now assume that \( A \) is projective and \( f: X \to X'' \) is an epimorphism. We must show that \( \text{Hom}(A, f) : \text{Hom}(A, X) \to \text{Hom}(A, X'') \) is also an epimorphism, i.e., that every \( h'' \in \text{Hom}(A, X'') \) admits a \( h \in \text{Hom}(A, X) \) such that \( fh = h'' \). However, this is just the property used in the definition of a projective object. This completes the proof of Lemma 12.1. \( \square \)

As pointed out in 11.3, the construction of right derived functors, applies to additive left exact functors in abelian categories with enough injective objects. Therefore, it can be applied to the functor \( \text{Hom}(A, .) : \text{Mod}^A \to \text{Mod} \), for any fixed inverse system \( A \in \text{Mod}^A \). It yields a sequence of additive functors, denoted by \( \text{Ext}^n(A, .) : \text{Mod}^A \to \text{Mod} \), \( n \geq 0 \). Moreover, for exact sequences
\[
E = (0 \to X' \stackrel{f'}{\to} X \stackrel{f}{\to} X'' \to 0),
\] (8)
it yields connecting homomorphisms \( \theta^E: \text{Ext}^n(A, X'') \to \text{Ext}^{n+1}(A, X') \). We now briefly repeat that construction, omitting details already given for \( \text{lim}^n \) in 11.3.

For \( X \in \text{Mod}^A \), one considers an injective resolution \( (I_X, e_X) \), where
\[
I_X = (0 \to I_0 \overset{d_1}{\longrightarrow} I_1 \overset{d_2}{\longrightarrow} I_2 \overset{d_3}{\longrightarrow} \cdots)
\] (9)
and \( e_X: X \to I_0 \) is a monomorphism. Then one considers the induced cochain complex
\[
\text{Hom}(A, I_X) = (0 \to \text{Hom}(A, I_0) \to \text{Hom}(A, I_1) \to \cdots).
\] (10)