20. Spectral sequences. Abelian groups

The proof of the key result of the next section (Theorem 21.6) uses in an essential way the Roos spectral sequence and its consequences, which we describe in this section (see 20.3). In order to make the text as self-contained as possible, we develop general techniques of spectral sequences in subsections 20.1 and 20.2. In 20.4 we discuss pure extensions of abelian groups and in 20.5 we establish the needed results from the theory of abelian groups.

20.1 The spectral sequence of a filtered complex

A (decreasing) filtration of a module $M$ is a sequence $F$ of submodules $M_p = F_p(M) \subseteq M, \ p \in \mathbb{Z}$, such that

\[ \ldots \supseteq M_p \supseteq M_{p+1} \supseteq \ldots, \]  
\[ \bigcup_p M_p = M. \]  
\[ (1) \]  
\[ (2) \]

The filtration $F$ is regular provided $M_p = 0$, for all sufficiently large $p$. Every filtered module $(M, F)$ determines a graded module, denoted by $G(M, F)$ and called the associated graded module of $(M, F)$. It consists of all the quotients $F_p(M)/F_{p+1}(M)$.

A (decreasing) filtration of a cochain complex $C = (C^n, \delta)$ is a sequence $F$ of subcomplexes $C_p = F_p(C) = (C^n_p, \delta)$ of $C$ such that, for each $n \in \mathbb{Z}$, the modules $C^n_p, \ p \in \mathbb{Z}$, form a filtration of the module $C^n$. The filtration of $C$ is regular provided the filtrations of all $C^n$ are regular.

A (regular) filtration $F$ of a cochain complex $C$ induces on each of the cohomology modules $H^n(C)$ a (regular) filtration $F$, defined by

\[ F_p(H^n(C)) = i_p(H^n(C_p)), \]  
\[ (3) \]

where $i_p$ is the homomorphism induced by the inclusion $C_p \rightarrow C$. Indeed, if $i_{p+1}: H^n(C_{p+1}) \rightarrow H^n(C_p)$ is the homomorphism induced by the inclusion $C_{p+1} \rightarrow C_p$, then $i_p i_{p+1} = i_{p+1}$ and thus,

\[ F_{p+1}(H^n(C)) = i_p i_{p+1}(H^n(C_{p+1})) \subseteq F_p(H^n(C)). \]  
\[ (4) \]

Moreover,
because every $n$-cocycle of $C = \bigcup C_p$ is also an $n$-cocycle of $C_p$, for some $p \in \mathbb{Z}$. If $F$ is regular on $C$, then, for any $n$, $C^n_p = 0$, provided $p$ is sufficiently large. Clearly, for such $p$ also $H^n(C_p) = 0$ and $F_p(H^n(C)) = 0$.

The filtration $F$ on $H^n(C)$, induced by the filtration $F$ on $C$, determines the associated bigraded cohomology module of $(C, F)$, which is denoted by $G(H^*(C), F)$. This is a bigraded module, i.e., a collection of modules indexed by pairs of integers. It consists of the modules

$$F_p(H^n(C)) / F_{p+1}(H^n(C)), \ p, q \in \mathbb{Z},$$

and contains valuable information on the cohomology modules $H^n(C)$, $n \in \mathbb{Z}$. However, it does not determine them completely, as the next example shows.

**Example 20.1.** Let $0 \rightarrow A \rightarrow B \rightarrow D \rightarrow 0$ and $0 \rightarrow A \rightarrow B' \rightarrow D \rightarrow 0$ be two exact sequences of abelian groups with $B'$ not isomorphic to $B$. E.g., such are the sequences $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0$ and $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}/2 \rightarrow \mathbb{Z}/2 \rightarrow 0$. Let $C$ be the cochain complex defined by $C^n = 0$, for $n \neq 0$, $C^0 = B$, so that $H^n(C) = 0$, for $n \neq 0$, and $H^0(C) = B$. Let $F$ be the filtration on $C$, defined by $C^n_p = 0$, for $n \neq 0$, and $C^0_p = B$, $A$, $0$, for $p < 0$, $p = 0$ and $p > 0$, respectively. Then $F_p(H^n(C)) = 0$, for $n \neq 0$, and $F_p(H^0(C)) = B$, $A$, $0$, for $p < 0$, $p = 0$ and $p > 0$, respectively. Consequently, the terms of the bigraded module of cohomology of $C$ equal 0, for $n \neq 0$, and equal 0, $B/A$, $A$, 0, for $p < -1$, $p = -1$, $p = 0$ and $p > 0$, respectively. Let $C'$ and $F'$ be defined analogously, using the second exact sequence instead of the first one. The bigraded cohomology modules of $C$ and $C'$ are isomorphic, because $B/A \approx B'/A$. Nevertheless, the cohomologies of $C$ and $C'$ differ, because $H^0(C) = B$ and $H^0(C') = B'$ are not isomorphic.

With every regularly filtered cochain complex one can associate a spectral sequence, which converges towards $G(H^*(C), F)$. To define spectral sequences, we need the notion of a bigraded differential module $(E, d)$ (see Fig. 20.1). This is a bigraded module $E = (E^{pq})$, endowed with a differential $d$ of bidegree $(r, -r + 1)$, for some $r \in \mathbb{Z}$, i.e., a collection of homomorphisms