ON THE LIE RING OF A GROUP OF PRIME EXPONENT

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Certain properties of a $p$-group $G$ are reflected in its Lie ring $L(G)$. For example, if the identity

$$x^p = 1$$

holds in $G$, then the identities

$$(1.1) \quad p^m = 0,$$

$$(1.2) \quad [\ldots [u, v], \ldots, v] = 0$$

hold in $L(G)$. Thus, we have an isomorphism of graded Lie rings

$$L(B(n)) \cong \Lambda(n)/\Sigma(n),$$

where $B(n)$ is the $n$-generator free group of the variety of groups defined by (1.1) and $\Lambda(n)$ the $n$-generator free Lie ring of the variety of Lie rings defined by (1.2) and (1.3).

Let $\Sigma_m(n)$ denote the homogeneous component of $\Sigma(n)$ of degree $m$. Sanov [8] proved that

$$\Sigma_m(n) = 0 \quad \text{for} \quad m \leq 2p-2$$

and asked whether $\Sigma(n) = 0$. Kostrikin [4] proved that

$$\Sigma_{2p-1}(2) = \Sigma_{2p}(2) = 0.$$ 

I will show, on the other hand, that

$$\Sigma_{2p-1}(n) \neq 0 \quad \text{when} \quad n \geq 3 \quad \text{and} \quad p = 5, 7, 11.$$ 

It seems likely that, in fact, (1.7) holds for all $p \geq 5$. Whether $\Sigma(2) = 0$ remains an open question.

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The component $\Sigma_m(n)$ has a naturally defined $\text{SL}(n, p)$-module structure. I prove that

\begin{equation}
(1.8) \quad \Sigma_{2p-1}(n) \text{ is either zero or an irreducible } \text{SL}(n, p)\text{-module of specified structure.}
\end{equation}

2. Preliminaries

It is largely a matter of taste whether one expresses the results below in terms of a Magnus algebra of formal power series or a suitable finite dimensional quotient algebra of it. We take the latter course.

Apart from supplying the necessary background material, the main purpose of the present section is to get a convenient representation of the group $B(n)$ - or rather, of $B(n, \sigma)$, its largest nilpotent quotient group of class $\leq \sigma$. This representation differs from those used by Magnus [6], Sanov [8] and Kostrikin [4] for similar purposes in that the coefficient domain is a field of characteristic $p$.

It is tacitly assumed that all associative algebras and rings introduced have unit elements. Homomorphisms, subalgebras, and so on, are interpreted accordingly.

2.1 THE ALGEBRA $A$

Let $k$ be a commutative ring and $n, \sigma$ positive integers. Then we denote by $A = A(n, \sigma; k)$ the associative $k$-algebra generated by (non-commuting) elements $x_1, \ldots, x_n$ subject to the following conditions:

(a) the monomials in $x_1, \ldots, x_n$ of total degree $\leq \sigma$ form a $k$-basis of $A$;

(b) all monomials of degree $> \sigma$ are zero.

Let $A^{(m)}$ denote the $k$-submodule of $A$ spanned by the monomials of total degree $m$. If $B$ is an additive subgroup of $A$, write $B^{(m)} = B \cap A^{(m)}$. We call $B$ graded when $B = \sum B^{(m)}$.

Clearly, every element of $A$ has a unique expansion

$$u = \sum u^{(m)}, \quad \{u^{(m)} \in A^{(m)}\}.$$ 

The first of the components $u^{(0)}, u^{(1)}, \ldots$ to be nonzero is called the leading term of $u$. If $S \subseteq A$, then we denote by $\text{gr} S$ the (graded) additive subgroup generated by the leading terms of the elements of $S$.

It is sometimes necessary to use the finer grading of $A$ by partial degrees.

Let $A^{(m_1, \ldots, m_n)}$ denote the $k$-submodule spanned by those monomials which have