Two-dimensional theory of the unsteady motion of fully cavitating hydrofoils

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1. Introduction

Hydrofoils, ship propellers and other hydrodynamic bodies, for which cavitation is a problem, have usually a slender shape and are operating at small angles with the direction of the main flow. It is therefore not unlikely to suppose that these cavity-flow problems admit of a linearized treatment. The first publication on a linearized cavity-flow problem, however, did not appear before 1953, when TuLIn [1] formulated a linearized theory for the symmetric flow about a cavitating wedge. Several other workers have contributed since to the systematic development of this linearized theory for steady cavity flows.

As to unsteady cavity-flow problems, however, the situation is quite different. It is true that the unsteady motion of a supercavitating hydrofoil presents no fundamental difficulties in the case of $\sigma = 0$, i.e., when the cavity is infinitely long (see Parkin [2]). Here $\sigma$ denotes the cavitation number defined by

$$\sigma = \frac{p_{\infty} - p_c}{\frac{1}{2} q U^2},$$

($p_{\infty}$: pressure at infinity; $p_c$: constant pressure of the cavity; $q$: density of the incompressible fluid; $U$: magnitude of the main stream velocity). In the case of $\sigma > 0$, however, when the cavity has finite length, some difficulties arise which are peculiar to unsteady cavity theory in two dimensions and which are not encountered in unsteady thin-airfoil theory.

2. Linearized theory of unsteady cavity flows in two dimensions

A typical condition used in the linearized theory of steady cavity flows of an incompressible fluid is the one requiring the cavity to be closed. This closure condition results in determining the unknown cavity length and is therefore of fundamental importance. It seems reasonable to extend this picture of the cavity as a closed vapour bubble to the linearized theory of unsteady cavity flows. Let us see what might be implied by this extension of the closure condition to unsteady cavity flows.

In the linearized theory of a fully cavitating hydrofoil in two dimensions, the area $A$ enclosed by hydrofoil plus cavity is given by

$$A = \int_0^1 \{f^+(x; t) - f^-(x; t)\} dx,$$
where \( y = f^\pm(x; t) \) represents the upper and lower side, respectively, of the boundary of the hydrofoil plus cavity (see Fig. 1). The quantity \( l \) will be called the cavity length.

During an unsteady motion of the hydrofoil the area of the cavity will generally change. Thus

\[
\frac{dA}{dt} = 0. \tag{2.2}
\]

Differentiating (2.1) with respect to time we obtain

\[
\frac{dA}{dt} = \int_0^l \left| \frac{\partial f^+}{\partial t} - \frac{\partial f^-}{\partial t} \right| dx + \frac{dl}{dt} \{ f^+(l; t) - f^-(l; t) \}. \tag{2.3}
\]

According to the closure condition the last term at the right-hand side of (2.3) vanishes. Using the linearized boundary condition

\[
v^\pm = \frac{\partial f^\pm}{\partial t} + \frac{\partial f^\pm}{\partial x}, \tag{2.4}
\]

in which \( v \) denotes the y-component of the perturbation velocity \( (u, v) \), and applying again the closure condition, we derive from (2.3) that

\[
\frac{dA}{dt} = \int_0^l \{ v^+ - v^- \} dx. \tag{2.5}
\]

It follows from (2.2) that generally

\[
\int_0^l \{ v^+ - v^- \} dx \neq 0. \tag{2.6}
\]

During a general unsteady motion, however, also

\[
\frac{d}{dt} \int_0^l \{ v^+ - v^- \} dx = 0. \tag{2.7}
\]

Using the second linearized Euler equation

\[
\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} = \frac{\partial \varphi}{\partial y}, \tag{2.8}
\]

in which \( \varphi(x; y; t) \) represents the acceleration potential defined by

\[
\varphi(x; y; t) = -\frac{p - Pe}{\varrho}, \tag{2.9}
\]

and introducing the harmonic conjugate \( \psi(x; y; t) \) of the acceleration potential we arrive at the following expression for the left-hand side of (2.7):

\[
\frac{d}{dt} \int_0^l \{ v^+ - v^- \} dx = -\{ \psi^+(l; t) - \psi^-(l; t) \} - \left\{ 1 - \frac{dl}{dt} \right\} \{ v^+(l; t) - v^-(l; t) \}. \tag{2.10}
\]

With a view to (2.7) we infer from (2.10) that in the case of a general unsteady motion at least one of the following two conditions must be satisfied:

i. The harmonic conjugate \( \psi(x; y; t) \) of the acceleration potential shows a discontinuity at the rear end of the cavity, viz.,

\[
\psi^+(l; t) - \psi^-(l; t) \neq 0, \tag{2.11}
\]

ii. The vertical component \( v(x; y; t) \) of the perturbation velocity shows a discontinuity at the rear end of the cavity, viz.,

\[
v^+(l; t) - v^-(l; t) \neq 0, \tag{2.12}
\]

whereas

\[
1 - \frac{dl}{dt} \neq 0. \tag{2.13}
\]