which is maximal among such entities, then \( f_i \) and \( F(P,t) \) are to admit continuous extensions to \( \Gamma \) which carry \( \Gamma \) into \( N \).

Then in the same notation as in Theorem 4.1 inequality (4.7) holds as do the same conclusions as to the consequences of equality in this inequality.

It follows at once that \( f_i \) is actually a homeomorphism into on the above arcs and boundaries and that \( f_i \) is regular thereon in terms of boundary uniformizers. Theorem 4.2 is most easily proved by forming the double \( \hat{\mathfrak{R}} \) of \( \mathfrak{R} \) across those maximal open boundary arcs and boundary components of \( \mathfrak{R} \) on which \( Q(z) \, dz^2 \leq 0 \). In this way we obtain a finite oriented Riemann surface which may have boundary components arising from boundary arcs and boundary components of \( \mathfrak{R} \) on which \( Q(z) \, dz^2 \geq 0 \).

As a Schottky differential \( Q(z) \, dz^2 \) extends to a quadratic differential on \( \hat{\mathfrak{R}} \) which we verify at once to be positive. From the families \( \{A\}, \{f\} \) and the deformation \( E \) we obtain in a natural manner similar entities on \( \hat{\mathfrak{R}} \) which are found to satisfy the conditions of Theorem 4.1 on the latter surface. Relative to \( \hat{\mathfrak{R}} \) the left hand side of (4.7) is replaced by a quantity just twice as large. Thus Theorem 4.1 applied to \( \hat{\mathfrak{R}} \) gives at once the corresponding inequality for \( \mathfrak{R} \). The results relative to equality extend immediately.

4.9 The exact status of condition (vii) in Definition 4.4 is not clear, that is, whether it is essential for the truth of Theorem 4.1 or merely for the success of the method used in its proof. It intervenes only in the evaluation of § 4.5 in the case of a strip domain. However the corresponding considerations for double poles, in particular the choice of the determination of \( \log a^{(0)} \), are unavoidable. It will be recalled that Royden [161] has recently stated a lemma [161, Lemma 1] which, if correct, would enable us to drop condition (vii) in Definition 4.4. It is a simple matter, however, to give counter examples to this lemma and an error is readily detected in its purported proof.

Chapter Five

Canonical Conformal Mappings

5.1 In this chapter we develop a number of standard canonical conformal mappings for domains of planar type and finite connectivity, giving at the end also some indications in the case of infinite connectivity. The method employs certain extremal properties of the canonical configurations together with compactness properties of the families of functions considered. Essentially the same approach has been earlier used by Grötzsch [65, 70] and Rengel [157] but here the use of the General Coefficient Theorem provides considerable unification and simplification.

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The first case treated is the following.

Theorem 5.1. Let the domain $D$ in the $z$-sphere be of finite connectivity and contain the origin and the point at infinity. Then there exists a unique function $\Phi(z)$ in $\Sigma_0(D)$ mapping $D$ conformally onto a domain bounded by slits on circles centred at the origin.

The results of the first seven theorems of this chapter are due to KoEBE [123, 124].

Lemma 5.1. Let the simply-connected domain $D$ contain the origin and the point at infinity. Then there exists a function in $\Sigma_0(D)$ mapping $D$ onto a domain bounded by a slit on a circle centred at the origin.

By the RIEMANN Mapping Theorem there exists a conformal mapping $w = f(z)$ of $D$ onto the exterior of $|w| = 1$ which carries the point at infinity into itself, the development there being

$$f(z) = az + \text{regular function of } z^{-1}$$

with $a$ not zero and such that $f(0) = -\varrho$, $\varrho$ real and greater than one. Let now

$$g(w) = \frac{aw^2 + \varrho^2w}{\varrho w + 1}.$$ 

Then it is readily verified that $a^{-1}g(f(z))$ provides the desired function.

Lemma 5.2. Let the domain $D$ in the $z$-sphere be bounded by a finite number of slits on circles centred at the origin. If $f \in \Sigma_0(D)$ then

$$|f'(0)| \leq 1.$$ (5.1)

Equality occurs in (5.1) only if $f(z) = z$.

We apply the General Coefficient Theorem, taking $\mathcal{R}$ to be the $z$-sphere, $Q(z) \, dz^2$ to be $-dz^2/z^2$, the family $\{A\}$ to consist of $D$ alone, $f$ to be the corresponding function. Evidently $D$ is an admissible domain. Also the function $f$ is admissible, the admissible homotopy existing automatically in this case and condition (vii) of Definition 4.4 being vacuously satisfied. The quadratic differential has double poles at the origin and the point at infinity which we denote by $P_1$ and $P_2$. In terms of local parameters at these points conditioned as in Theorem 4.1 we have the coefficients

$$\alpha^{(1)} = \alpha^{(2)} = -1$$

$$a^{(1)} = 1/|f'(0)|$$

$$a^{(2)} = 1.$$

Inequality (4.7) gives

$$\mathcal{R} \{ - \log |f'(0)|^{-1} \} \leq 0$$

the values of the deformation degree at $P_1$ and $P_2$ being inessential. This proves inequality (5.1). The equality statement comes under possibility (ii) of Theorem 4.1.