Some interesting extensions of these canonical mappings and their extremal properties have been given by Komatu and Ozawa [128]. Finally let it be remarked that most of these results are special cases of more general theorems which can be obtained by a general method recently given by the author [112].

Chapter Six

Applications of the General Coefficient Theorem.

Univalent Functions

6.1 We are now ready to apply the General Coefficient Theorem (Theorem 4.1) to prove many of the standard results on univalent functions. To be sure in many of these, particularly the most elementary ones, there is no mention of homotopy conditions corresponding to those which appear in the General Coefficient Theorem. The reason is that in these cases the homotopy conditions are automatically satisfied. Indeed we have

Lemma 6.1. Let \( \mathcal{R} \) be a simply-connected finite oriented Riemann surface, \( Q(z) \, dz^2 \) a quadratic differential on \( \mathcal{R} \) and \( \{ \Delta \} \) an admissible family of domains \( \Delta_j, j = 1, \ldots, k \), on \( \mathcal{R} \) with respect to \( Q(z) \, dz^2 \) such that each domain \( \Delta_j \) is simply-connected. Let there be \( M_j \) (distinct) poles of \( Q(z) \, dz^2 \) interior to \( \Delta_j \). Let \( N_j \) be one if \( \mathcal{R} \) has a boundary or if there is a pole of \( Q(z) \, dz^2 \) exterior to \( \Delta_j \). Let

\[
M_j + N_j \leq 3, \quad j = 1, \ldots, k.
\]

Let \( \{ f \} \) be a family of functions \( f_j, j = 1, \ldots, k \), with the properties (i), (ii), (iii) of Definition 4.2, and such that further no pole of \( Q(z) \, dz^2 \) not interior to \( \bigcup_{j=1}^{k} \Delta_j \) is interior to \( \bigcup_{j=1}^{k} f_j(\Delta_j) \). Then the family \( \{ f \} \) admits an admissible homotopy into the identity.

This lemma follows from well known results on the homotopy properties of simply-connected, doubly-connected and triply-connected domains [119, 26].

6.2 Theorem 6.1. For \( f \in S \) the image of \( |z| < 1 \) under the mapping \( w = f(z) \) covers the circle \( |w| < 1/4 \). It omits a point with \( |w| = 1/4 \) only if \( f(z) = z(1 + e^{i\alpha} z)^{-2} \), \( \alpha \) real.

We apply the General Coefficient Theorem with \( \mathcal{R} \) the \( w \)-sphere,

\[
Q(w) \, dw^2 = \frac{dw^2}{w^k(w - 1/4)},
\]

and a single admissible domain \( \Delta = \hat{f}(E) \) where \( E \) is \( |z| < 1 \) and \( \hat{f}(z) = z(1 + z)^{-2} \). Let \( \Phi(w) \) be the inverse of \( \hat{f} \) defined on \( \Delta \). Suppose
that \( f(E) \) does not cover the point \( 1/4 \), \( \alpha \) real. Then \( e^{i\alpha} f(e^{-i\alpha}z) \) is in \( S \) and \( e^{i\alpha} f(e^{-i\alpha}E) \) does not cover the point \( 1/4 \). Then the function
\[
\Psi(w) = e^{i\alpha} f(e^{-i\alpha} \Phi(w))
\]
is seen to be an admissible function associated with \( \Delta \) (Definition 4.4 and Lemma 6.1).

The quadratic differential has a single double pole \( P_1 \) at the origin. The corresponding coefficients are

\[
\alpha^{(1)} = -4, \quad a^{(1)} = 1.
\]

Thus equality occurs in inequality (4.7) (the determination of log \( a^{(1)} \) being inessential). By equality condition (ii) in Theorem 4.1
\[
e^{i\alpha} f(e^{-i\alpha} \Phi(w)) = w, \quad w \in \tilde{f}(E).
\]
Setting \( \Phi(w) = e^{i\alpha}z \) we obtain
\[
f(z) = z \left(1 + e^{i\alpha}z\right)^{-2}, \quad |z| < 1
\]
as stated.

Theorem 6.2. Let \( f \in S \). Then
\[
\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, \quad |z| = r, \quad 0 < r < 1
\]
equality occurring on each side only for \( f(z) = z \left(1 + e^{i\alpha}z\right)^{-2}, \alpha \) real, respectively for \( z = re^{-i\alpha} \) and \( z = -re^{-i\alpha} \).

Of course the right hand inequality holds for \( |z| \leq r \).

To prove the lower bound we let \( \Re \) and \( \Delta \) be as in Theorem 6.1. We set \( d = r(1+r)^{-2}, \quad c = f(re^{i\theta}), \quad \theta \) real. Then we choose
\[
Q(w) \, dw^2 = \frac{dw^2}{w^2(w-d)}
\]
and with \( \Phi \) as in Theorem 6.1 we take
\[
g(w) = \frac{d}{c} f(e^{i\theta} \Phi(w)) .
\]
It is verified at once that \( \Delta \) is an admissible domain with respect to \( Q(w) \, dw^2 \) and that \( g \) is an admissible function associated with \( \Delta \).

The quadratic differential has a single double pole \( P_1 \) at the origin. The corresponding coefficients are

\[
\alpha^{(1)} = -d^{-1}, \quad a^{(1)} = c/de^{i\theta}.
\]
Inequality (4.7) becomes
\[
\Re \{ -d^{-1} \log(c/de^{i\theta}) \} \leq 0
\]
which easily reduces to
\[
|f(re^{i\theta})| \geq \frac{r}{(1+r)^2} .
\]
By equality condition (iii) in Theorem 4.1 equality can occur here only if
\[
\frac{d}{c} f(e^{i\theta} \Phi(w)) = w, \quad w \in \tilde{f}(E) .
\]