Projective Algebraic Geometry

So far, all of the varieties we have studied have been subsets of affine space \( k^n \). In this chapter, we will enlarge \( k^n \) by adding certain "points at \( \infty \)" to create \( n \)-dimensional projective space \( \mathbb{P}^n(k) \). We will then define projective varieties in \( \mathbb{P}^n(k) \) and study the projective version of the algebra–geometry correspondence. The relation between affine and projective varieties will be considered in §4; in §5, we will study elimination theory from a projective point of view. By working in projective space, we will get a much better understanding of the Extension Theorem from Chapter 3. The chapter will end with a discussion of the geometry of quadric hypersurfaces and an introduction to Bezout's Theorem.

§1 The Projective Plane

This section will study the projective plane \( \mathbb{P}^2(\mathbb{R}) \) over the real numbers \( \mathbb{R} \). We will see that, in a certain sense, the plane \( \mathbb{R}^2 \) is missing some "points at \( \infty \)," and by adding them to \( \mathbb{R}^2 \), we will get the projective plane \( \mathbb{P}^2(\mathbb{R}) \). Then we will introduce homogeneous coordinates to give a more systematic treatment of \( \mathbb{P}^2(\mathbb{R}) \).

Our starting point is the observation that two lines in \( \mathbb{R}^2 \) intersect in a point, except when they are parallel. We can take care of this exception if we view parallel lines as meeting at some sort of point at \( \infty \). As indicated by the picture at the top of the following page, there should be different points at \( \infty \), depending on the direction of the lines. To approach this more formally, we introduce an equivalence relation on lines in the plane by setting \( L_1 \sim L_2 \) if \( L_1 \) and \( L_2 \) are parallel. Then an equivalence class \([L]\) consists of all lines parallel to a given line \( L \). The above discussion suggests that we should introduce one point at \( \infty \) for each equivalence class \([L]\). We make the following provisional definition.

**Definition 1.** The projective plane over \( \mathbb{R} \), denoted \( \mathbb{P}^2(\mathbb{R}) \), is the set

\[
\mathbb{P}^2(\mathbb{R}) = \mathbb{R}^2 \cup \{ \text{one point at } \infty \text{ for each equivalence class of parallel lines}\}.
\]
We will let \( [L]_\infty \) denote the common point at \( \infty \) of all lines parallel to \( L \). Then we call the set \( \overline{L} = L \cup [L]_\infty \subset \mathbb{P}^2(\mathbb{R}) \) the \textit{projective line} corresponding to \( L \). Note that two projective lines always meet in exactly one point: if they are not parallel, they meet at a point in \( \mathbb{R}^2 \); if they are parallel, they meet at their common point at \( \infty \).

At first sight, one might expect that a line in the plane should have two points at \( \infty \), corresponding to the two ways we can travel along the line. However, the reason why we want only one is contained in the previous paragraph: if there were two points at \( \infty \), then parallel lines would have two points of intersection, not one. So, for example, if we parametrize the line \( x = y \) via \((x, y) = (t, t)\), then we can approach its point at \( \infty \) using either \( t \to \infty \) or \( t \to -\infty \).

A common way to visualize points at \( \infty \) is to make a perspective drawing. Pretend that the earth is flat and consider a painting that shows two roads extending infinitely far in different directions:

For each road, the two sides (which are parallel, but appear to be converging) meet at the same point on the horizon, which in the theory of perspective is called a \textit{vanishing point}. Furthermore, any line parallel to one of the roads meets at the same vanishing point, which shows that the vanishing point represents the point at \( \infty \) of these lines.