(13.4) If $X$ is a finite CW complex with a weak $n$-dual $D_nX$, there is a canonical isomorphism

$$u' : \Omega_k(X) \cong \widetilde{\Omega}^{n-k-1}(D_nX).$$

**Proof.** We may confine ourselves to finite simplicial complexes $X$ embedded as proper subcomplexes of $S^n$. Since $X$ is contractible to a point in $S$ there is a short exact sequence $0 \to \Omega^{n-k-1}(X) \to \Omega^n(S^n, X) \to \widetilde{\Omega}^{n-k}(S^n) \to 0$. There is also the exact sequence

$$0 \to \Omega_k(S^n \setminus X) \to \Omega_k(S^n \setminus X) \to \Omega_k(S^n) \to 0.$$ Duality yields a diagram

$$0 \to \Omega_k(S^n \setminus X) \to \Omega_k(S^n \setminus X) \to \Omega_k \to 0.$$ There is then a unique isomorphism $\Omega_k(S^n \setminus X) \cong \Omega^{n-k-1}(S^n, X)$ such that commutativity holds. Since $\Omega_k(D_nX) = \Omega_k(S^n \setminus X)$, we get an isomorphism $u' : \Omega_k(D_nX) \cong \Omega^{n-k-1}(X)$, which is sufficient to show (13.4).

**CHAPTER II**

**Computation of the bordism groups**

In the previous chapter, we have defined and characterized geometrically the homology theory $\{\Omega_*(X, A), \varphi_*, \partial\}$ of bordism. Thus the stage is set for their computation, at least in many cases. In order to compute, we use the powerful results of MILNOR ($\Omega_*$ has no odd torsion) and WALL (see section 14) on $MSO(k)$. In section 14 we prove that the bordism spectral sequence is trivial modulo the class of odd torsion groups. In section 15 it is proved that if $X$ has no odd torsion then $\Omega_*(X) = \Sigma_{p+q=n} H_p(X; \Omega_q)$; in section 18 it is shown that if $X$ has no torsion then $\Omega_*(X) \cong H_*(X; \mathbb{Z}) \otimes \Omega$ as an $\Omega$-module.

Generalizing the Stiefel-Whitney numbers and the Pontryagin numbers of a manifold, in section 17 we define natural numerical invariants of maps $f : M^n \to X$. These are functions only of the bordism class of $f$. If all torsion of $X$ consists of elements of order two, the bordism class of $f$ is determined by the Whitney numbers and the Pontryagin numbers of $f$.

**14. Triviality mod C**

Denote by $C$ the class of torsion groups having all elements of odd order. The fundamental result of this chapter is the following.

**Theorem.** For any CW pair $(X, A)$ the bordism spectral sequence is trivial mod $C$. 

---

P. E. Conner et al., *Differentiable Periodic Maps*  
© Springer-Verlag Berlin Heidelberg 1964
The purpose of this section is to prove (14.1). We must show that the image of each $d^r_p: E^r_{p,q} \rightarrow E^r_{p-r,q+r-1}$ is an odd torsion group. We use in a basic way the following theorem of C. T. C. WALL [42], which is now assumed.

(14.2) Wall. The module $H^*(MSO(k); \mathbb{Z}_2)$ over the mod 2 Steenrod algebra is isomorphic in dimensions $<2k$ to a direct sum of Steenrod algebras $H^*(Z, m_i; \mathbb{Z}_2)$ and $H^*(Z, n_i; \mathbb{Z}_2)$.

Put in terms of spectra of § 12, $H^*(MSO; \mathbb{Z}_2)$ is isomorphic as a module over the Steenrod algebra to a direct sum of copies of $H^*(K(Z); \mathbb{Z}_2)$ and of $H^*(K(Z); \mathbb{Z}_2)$.

Proof of (14.1). Let $a \in H^m_{1}(MSO; \mathbb{Z}_2)$ denote the generator of one of the submodules of $H^*(MSO; \mathbb{Z}_2)$ isomorphic to $H^*(K(Z); \mathbb{Z}_2)$. The Bockstein $S_1^1: H^m_{1}(MSO; \mathbb{Z}_2) \rightarrow H^{m+1}_{1}(MSO; \mathbb{Z}_2)$ kills $a$. Hence the integral Bockstein $H^m_{1}(MSO; \mathbb{Z}_2) \rightarrow H^{m+1}_{1}(MSO; \mathbb{Z})$ maps $a$ into an element $a', a'$ of order two, which is zero when restricted mod 2. Hence $a' = 2b$ for some $b \in H^{m+1}_{1}(MSO; \mathbb{Z}_2)$. But additively $H^*(MSO(k); \mathbb{Z}_2) \cong \cong H^*(BSO(k); \mathbb{Z}_2)$ by the Thom isomorphism (11.2), and hence all 2-torsion consists of elements of order two by the results in § 10. Since $2a' = 0$, $a' = 2b$ it follows that $a' = 0$. Thus $a$ is the restriction of an integral class $a \in H^m_{1}(MSO; \mathbb{Z}_2)$.

The elements $a$, $a$ are represented by unique elements $\alpha_k \in H^k_{m_i}(MSO(k); \mathbb{Z}_2)$ and $a_k \in H^k_{m_i}(MSO(k); \mathbb{Z}_2)$ for $k > m_i$. For each $k > m_i$ there is a cellular map $I_k: MSO(k) \rightarrow K(Z, m_i + k)$, unique up to homotopy, with $I_k(\tau) = \alpha_k$, $\tau \in H^{m_i+k}(Z, m_i + k; \mathbb{Z}_2)$ the fundamental class. Also $I_k(\text{mod } 2) = a_k$.

The diagram

$$
\begin{array}{c}
SMSO(k) \longrightarrow MSO(k + 1) \\
\downarrow S_l \\
SK(Z, m_i + k) \longrightarrow K(Z, m_i + k + 1),
\end{array}
$$

where the horizontal maps are the spectrum maps, is then seen to be commutative up to homotopy.

Consider now a variant of the homology theory of § 12 based on the spectrum $K(Z)$. Define

$$iK_s(X, A) = \text{Dir Lim}_s \pi_{s+k}(X/A \wedge K(Z, m_i + k)).$$

It follows from § 12 that $iK_s(X, A) = H_s-m_i(X, A; \mathbb{Z}_2)$. There is a spectral sequence $\{iE_{p,q}^r\}$ for the homology theory $iK_s(X, A)$. We have $iE_{p,q}^1 = iK_{p+q}(X^p/X^{p-1}) \cong H_{p+q-m_i}(X^p, X^{p-1}; \mathbb{Z}_2)$. Hence $iE_{p,q}^1 = 0$ if $q \neq m_i$, $iE_{p,m_i} = C_p(X, A)$. It is also the case that $iE_{p,m_i} = H_s(X, A; \mathbb{Z}_2)$. Since there is just one non-zero fiber degree, the spectral sequence is trivial for $r \geq 2$. The maps $I_k: MSO(k) \rightarrow K(Z, m_i + k)$, all $k$, induce