Fixed points of maps of odd prime period

We consider now the fixed point sets of differentiable maps $T : M^n \to M^n$ of odd prime period $p$. In section 38 we analyze the normal bundle $\xi$ to the fixed point set. It turns out to be as simple as could be expected, breaking into a Whitney sum of complex vector bundles $\xi_k$ on which $T$ acts as multiplication by $\varrho^k$, $\varrho = \exp(2\pi i/p)$.

We then illustrate some connections between the properties of such $T : M^n \to M^n$ and the structure of $\Omega^* (Z_p)$ found in Chapter VII. In section 39 we describe those bordism classes in $\Omega$ which admit representatives upon which $Z_p \times Z_p$ acts differentiably, preserving orientation, and with no stationary points. In section 40 we study those $T : M^n \to M^n$ whose fixed point sets are $m$-manifolds, $F^m$, with trivial normal bundles in an appropriate sense. There follows a purely topological section giving the structure of the ideal in $\Omega$ consisting of all those $[M^n]$ whose Pontryagin numbers are all divisible by $p$. We then return in section 42 to the study of those $T$ for which the normal bundle to the fixed point is trivial; here we obtain additional insight into the module structure of $\Omega^* (Z_p)$.

38. Generalities about the normal bundle

Let $H$ be a compact Lie group which acts differentiably on a manifold $M^n$. There is a Riemannian metric on $M^n$ with respect to which $H$ acts as a group of isometries. Denote by $F^m$ the union of the $m$-dimensional components of the set of stationary points. There is the normal bundle $\xi : E \to F^m$ to $F^m$ in $M^n$, and $\xi$ can be thought of as an $O(n-m)$-bundle. Moreover, $H$ acts on $E$ as a group of bundle maps, mapping each fibre into itself. If we fix a fibre $V_x$ of $\xi$, then on $V_x$ we have a linear representation of $H$ and thus a (non-unique) embedding $H \subset O(n-m)$. We shall prove in this section a theorem which implies that the structural group of $\xi$ can be reduced, on the component containing $x$, to the centralizer of $H$ in $O(n-m)$. We go on to consider the case $H = Z_p$ in detail. To handle the non-abelian case we need the following mild extension of a well known Montgomery-Zippin theorem [29, p. 216].

(38.1) Lemma. Let $r_0 : H \to G$ be a homomorphism of the compact Lie group $H$ into the Lie group $G$. For each homomorphism $r : H \to G$ sufficiently close to $r_0$, there exists $g \in G$ with $r = g r_0 g^{-1}$.

Proof. Consider the Lie group $H \times G$ and the graph $K(r_0) \subset H \times G$ of $r_0$, where $K(r_0) = \{(x, r_0(x)) : x \in H\}$. Since $K(r_0)$ is a compact subgroup of $H \times G$, the Montgomery-Zippin result asserts that there exists a neighborhood $U$ of $K(r_0)$ such that if $K'$ is a closed subgroup of $H \times G$ with $K' \subset U$ then $g K' g^{-1} \subset K(r_0)$ for some $g \in H \times G$. Suppose for
$r: H \to G$ that $K(r) \subset U$. Let $(h, g) \in H \times G$ be such that $(h, g) \cdot K(r) \times (h, g)^{-1} \subset K(r_0)$. For each $x \in H$ there is $y \in H$ with $(h \cdot x \cdot h^{-1}, g \cdot r(x) \cdot g^{-1}) = (y, r_0(y))$ and $g \cdot r(x) \cdot g^{-1} = r_0(h) \cdot r_0(h^{-1})$, so (38.1) follows.

We now consider fiber bundles $\xi: E \to X$ which are co-ordinate bundles in the sense of STEENROD, and for which the base is connected, locally connected and paracompact. We suppose that the structural group $G$ is a compact Lie group which acts effectively on the fiber $F$, and $F$ is to be locally compact. We may consider $G$ as a subgroup of the group of homeomorphisms of $F$ onto itself.

(38.2) Theorem. Let $\xi: E \to X$ be a fiber bundle with structural group $G$ and fiber $F$ as above. Let $H$ be a compact Lie group which acts on $E$ as a group of bundle maps, taking each fiber effectively onto itself. Then the structural group of $\xi$ can be reduced to the centralizer $C(H')$ of $H'$ in $G$, where $H' \subset G$ is the subgroup of homeomorphisms of $F$ corresponding to $H$ under some coordinate transformation $F \to F$.

Proof. Let $(U_i, \varphi_i)$ be a coordinate set for $\xi$; that is, $U_i$ is open in $X$ and $\varphi_i: U_i \times F \to E$ has the usual properties. For $x \in U_i$, let $\varphi_{i, x}: F_x \to F$ denote the homeomorphism $\varphi_{i, x}(f) = \varphi_i(x, f)$. For $h \in H$, we let $h_{i, x} = \varphi_{i, x}^{-1} h \varphi_{i, x}$ and $H_{i, x} = \{h_{i, x}: h \in H\}$. Since each $h$ acts as a bundle map, then $H_{i, x} \subset G$. Let $r_{i, x}: H \to G$ denote the homomorphism $r_{i, x}(h) = h_{i, x}$.

If $x \in U_i \cap U_j$ then there is a $g \in G$ for which $r_{j, x} = g^{-1} r_{i, x} g$, for $h_{i, x} = \varphi_{i, x}^{-1} h \varphi_{i, x} = (\varphi_{i, x}^{-1} \varphi_{i, x}) (\varphi_{i, x}^{-1} h \varphi_{i, x}) (\varphi_{i, x}^{-1} \varphi_{j, x}) = g_{i, j}^{-1}(x) \cdot h_{i, x} \cdot g_{i, j}(x)$, where $g_{i, j}(x) = \varphi_{i, x}^{-1} \varphi_{j, x}$.

We consider now a coordinate neighborhood $(U_i, \varphi_i)$ where $U_i$ is connected. Fix $x_0 \in U_i$, and consider the subset $V \subset U_i$ consisting of all $x$ for which there exists a $g \in G$ with $r_{i, x} = g r_{i, x_0} g^{-1}$. It is clear that $V$ is closed in $U_i$; it follows from (38.1) that it is also open in $U_i$. Hence for any $x \in U_i$ there exists $g \in G$ with $r_{i, x} = g r_{i, x_0} g^{-1}$. It now follows from the connectedness of $X$ that if $x \in U_i$ and $y \in U_j$ then there exists $g \in G$ with $r_{j, y} = g r_{i, x} g^{-1}$. We fix a homomorphism $r: H \to G$ for which each $r_{i, x}$ is conjugate in $G$ to $r$.

There is the space $Y$ of all homomorphisms of $H$ into $G$ which have the form $g r g^{-1}$. The space $Y$ is naturally homeomorphic to $G / C(H')$, where $H' = \text{Image}(r)$ and $C(H')$ is the centralizer of $H'$ in $G$. For a given connected coordinate neighborhood $U_i \subset X$, there is the continuous map $f: U_i \to Y$ mapping $x \in U_i$ into $r_{i, x} \in Y$. There is the map $G \to Y$ sending $g$ into $g r g^{-1}$. Since $Y \cong G / C(H')$, the map $G \to Y$ has a local cross-section. If $x_0 \in U_i$ there is a connected neighborhood $V_j \subset U_i$ with $x_0 \in V_j$ and a map $x \to g_x$ of $V_j$ into $G$ with $g_x r x g_x^{-1} = r_{i, x}$ for each $x \in V_j$. We define $\theta_j: V_j \times F \to \xi^{-1}(V_x)$ by $\theta_j(x, f) = \varphi_i(x, g_x(f))$; then