On the Broader Epistemological Significance of Self-Justifying Axiom Systems

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Abstract. This article will be a continuation of our research into self-justifying systems. It will introduce several new theorems (one of which will transform our previous infinite-sized self-verifying logics into formalisms or purely finite size). It will explain how self-justification is useful, even when the Incompleteness Theorem clearly limits its scope.

1 Introduction

Gödel’s Incompleteness Theorem has two parts. Its first half indicates no decision procedure can identify arithmetic’s true statements. Its “Second Incompleteness” result specifies sufficiently strong logics cannot verify their own consistency. Gödel was careful to insert a caveat into his historic paper [11], indicating a diluted form of Hilbert’s Consistency Program might have some success:

∗ “It must be expressly noted Proposition XI represents no contradiction of the formalistic standpoint of Hilbert. For this standpoint presupposes only the existence of a consistency proof by finite means, and there might conceivably be finite proofs which cannot be stated in P or in ...”

Some scholars have interpreted ∗ as, possibly, anticipating attempts to confirm Peano Arithmetic’s consistency, via either Gentzen’s formalism or Gödel’s Dialectica interpretation. On the other hand, the Stanford’s Encyclopedia’s entry about Gödel quotes him, in its Section 2.2.4, stating he was hesitant to view the Second Incompleteness Theorem as fully ubiquitous, until learning of Turing’s work. Moreover, Yourgrau [45] states von Neumann “argued against Gödel himself” in the early 1930’s, about the definitive termination of Hilbert’s consistency program, which “for several years” after [11]’s publication, Gödel “was cautious not to prejudge”. Also, it is known [6,13,45] that Gödel did initially presume the second theorem was false, before proving its stunning result.

In any case several year after he wrote ∗’s initial statement, Gödel gave a 1933 lecture [12], where he told his audience that Hilbert’s initial 1926 objectives, summarized formally by ∗∗ below, had “unfortunately” no “hope of succeeding along” its originally intended plans.

∗∗ (Hilbert [17] 1926): “Where else would reliability and truth be found if even mathematical thinking fails? The definitive nature of the infinite has become necessary, not merely for the special interests of individual sciences, but rather for the honor of human understanding itself.”
Our research, in both the current article and prior papers \cite{35-44}, was stimulated by the prospect that we find ** enticing, even though the Second Incompleteness Theorem unequivocally demonstrates that logics cannot recognize their own consistency in a robust sense. Accordingly, we have studied both generalizations and boundary-case exceptions for the Second Incompleteness Theorem in \cite{35-44}. The current article will seek to both strengthen these prior results, in the context of axiom systems with strictly finite cardinalities, and to also provide a more intuitive explanation of the meaning behind \cite{35-44}’s results.

The thesis of this article will be delicate because there can be no doubt that the Second Incompleteness Theorem is sharply robust, when viewed from a conventional purist mathematical perspective. On the other hand, we will argue that there are certain facets of a “Self-Justifying Logics”, that are tempting under a hard-nosed engineering perspective, contemplating sharply curtailed forms of Hilbert’s goals. These results will be fragile but not fully immaterial.

2 Background Setting

Let \((\alpha, d)\) denote any axiom system and deduction method satisfying the simple “Split Rule” below. This pair will be called “Self Justifying” when:

i. one of \(\alpha\)’s theorems will state that the deduction method \(d\), applied to the system \(\alpha\), will produce a consistent set of theorems, and

ii. the axiom system \(\alpha\) is in fact consistent.

For any \((\alpha, d)\), it is easy to construct a second \(\alpha^d \supseteq \alpha\) that satisfies the Part-i requirement. For instance, \(\alpha^d\) could consist of all of \(\alpha\)’s axioms plus an added “SelfRef(\(\alpha, d\))” sentence, defined as stating:

- There is no proof (using \(d\)’s deduction method) of \(0 = 1\) from the union of the system \(\alpha\) with this sentence “SelfRef(\(\alpha, d\))” (looking at itself).

Kleene \cite{20} noted how to encode rough analogs of “SelfRef(\(\alpha, d\))”. Each of Kleene, Rogers and Jeroslow \cite{19,20,29} noted \(\alpha^d\) may, however, be inconsistent (despite SelfRef(\(\alpha, d\))’s assertion), thus causing it to violate Part-ii’s requirement.

This problem arises in many contexts besides Gödel’s paradigm, where \(\alpha\) was an extension of Peano Arithmetic (see \cite{1,5,7,9,11,14,16,18,21,23,25,26,34,38,39,43}). Such results formalize paradigms where self-justification is infeasible, due to diagonalization issues. (It should, perhaps, be added that among this lengthy list of articles, it was especially \cite{1,4,11,23,27,31,34}’s incompleteness results that influenced our work in \cite{35,44}.) In any case, the main point is that most logicians have hesitated to employ an analog of a SelfRef(\(\alpha, d\)) axiom because \(\alpha^d = \alpha + \text{SelfRef}(\alpha, d)\) is typically inconsistent.

\footnote{Our “Split Rule” is the trivial requirement that all the axiom sentences in \(\alpha\) are technically proper axioms, and that deduction method \(d\) is required to include BOTH a finite number of rules of inference and whatever “logical axioms” are needed (if any ? ) by \(d\)’s methodology. (This trivial notation convention is helpful.)}