Three applications of Euler’s formula

A graph is planar if it can be drawn in the plane \( \mathbb{R}^2 \) without crossing edges (or, equivalently, on the 2-dimensional sphere \( S^2 \)). We talk of a plane graph if such a drawing is already given and fixed. Any such drawing decomposes the plane or sphere into a finite number of connected regions, including the outer (unbounded) region, which are referred to as faces. Euler’s formula exhibits a beautiful relation between the number of vertices, edges and faces that is valid for any plane graph. Euler mentioned this result for the first time in a letter to his friend Goldbach in 1750, but he did not have a complete proof at the time. Among the many proofs of Euler’s formula, we present a pretty and “self-dual” one that gets by without induction. It can be traced back to von Staudt’s book “Geometrie der Lage” from 1847.

**Euler’s formula.** If \( G \) is a connected plane graph with \( n \) vertices, \( e \) edges and \( f \) faces, then

\[
n - e + f = 2.
\]

**Proof.** Let \( T \subseteq E \) be the edge set of a spanning tree for \( G \), that is, of a minimal subgraph that connects all the vertices of \( G \). This graph does not contain a cycle because of the minimality assumption.

We now need the dual graph \( G^* \) of \( G \): To construct it, put a vertex into the interior of each face of \( G \), and connect two such vertices of \( G^* \) by edges that correspond to common boundary edges between the corresponding faces. If there are several common boundary edges, then we draw several connecting edges in the dual graph. (Thus \( G^* \) may have multiple edges even if the original graph \( G \) is simple.)

Consider the collection \( T^* \subseteq E^* \) of edges in the dual graph that corresponds to edges in \( E \setminus T \). The edges in \( T^* \) connect all the faces, since \( T \) does not have a cycle; but also \( T^* \) does not contain a cycle, since otherwise it would separate some vertices of \( G \) inside the cycle from vertices outside (and this cannot be, since \( T \) is a spanning subgraph, and the edges of \( T \) and of \( T^* \) do not intersect). Thus \( T^* \) is a spanning tree for \( G^* \).

For every tree the number of vertices is one larger than the number of edges. To see this, choose one vertex as the root, and direct all edges “away from the root”: this yields a bijection between the non-root vertices and the edges, by matching each edge with the vertex it points at. Applied to the tree \( T \) this yields \( n = e_T + 1 \), while for the tree \( T^* \) it yields \( f = e_{T^*} + 1 \). Adding both equations we get \( n + f = (e_T + 1) + (e_{T^*} + 1) = e + 2 \). □
Euler’s formula thus produces a strong numerical conclusion from a geometric-topological situation: the numbers of vertices, edges, and faces of a finite graph $G$ satisfy $n - e + f = 2$ whenever the graph is or can be drawn in the plane or on a sphere.

Many well-known and classical consequences can be derived from Euler’s formula. Among them are the classification of the regular convex polyhedra (the platonic solids), the fact that $K_5$ and $K_{3,3}$ are not planar (see below), and the five-color theorem that every planar map can be colored with at most five colors such that no two adjacent countries have the same color. But for this we have a much better proof, which does not even need Euler’s formula — see Chapter 38.

This chapter collects three other beautiful proofs that have Euler’s formula at their core. The first two — a proof of the Sylvester–Gallai theorem, and a theorem on two-colored point configurations — use Euler’s formula in clever combination with other arithmetic relationships between basic graph parameters. Let us first look at these parameters.

The degree of a vertex is the number of edges that end in the vertex, where loops count double. Let $n_i$ denote the number of vertices of degree $i$ in $G$. Counting the vertices according to their degrees, we obtain

$$n = n_0 + n_1 + n_2 + n_3 + \cdots \quad (1)$$

On the other hand, every edge has two ends, so it contributes 2 to the sum of all degrees, and we obtain

$$2e = n_1 + 2n_2 + 3n_3 + 4n_4 + \cdots \quad (2)$$

You may interpret this identity as counting in two ways the ends of the edges, that is, the edge-vertex incidences. The average degree $\overline{d}$ of the vertices is therefore

$$\overline{d} = \frac{2e}{n}.$$ 

Next we count the faces of a plane graph according to their number of sides: a $k$-face is a face that is bounded by $k$ edges (where an edge that on both sides borders the same region has to be counted twice!). Let $f_k$ be the number of $k$-faces. Counting all faces we find

$$f = f_1 + f_2 + f_3 + f_4 + \cdots \quad (3)$$

Counting the edges according to the faces of which they are sides, we get

$$2e = f_1 + 2f_2 + 3f_3 + 4f_4 + \cdots \quad (4)$$

As before, we can interpret this as double-counting of edge-face incidences. Note that the average number of sides of faces is given by

$$\overline{f} = \frac{2e}{f}.$$