The “Borromean rings” — three rings arranged so that no two of them are linked, but the configuration cannot be taken apart without breaking one of the rings — form a classic artistic symbol, which appeared in the coat of arms of the aristocratic Borromeo family since the middle of the 15th century.

The Borromean rings are also one of the most tantalizing and enigmatic “impossible figures” of mathematics. They can easily be built as a geometric object in such a way that two of the rings are perfectly round circles of the same size; it seems, however, that then the third ring is represented by an ellipse, at best. Thus it is natural to ask:

**Can the Borromean rings be built from three perfect circles?**

As mathematical objects, the Borromean rings belong to the theory of knots and links, which very attractively connects geometry, topology, and combinatorics. We all have a geometric picture of what knots (closed curves in space) and links (arrangements of several such curves) look like, and we can draw them in the plane. We also have intuitive notions of when two knots or links are “the same” (equivalent), when a knot or link is “trivial,” when two circles are linked, etc.: The appendix to this chapter provides a review of the essential terms and definitions, including the fact that two diagrams present the same link or knot if and only if they can be transformed into each other by a finite sequence of “Reidemeister moves.”

Knot theory as we know it today started in 1867, when the physicist William Thompson, now known as Lord Kelvin, came up with his “vortex theory,” according to which atoms could be explained as knots in the “ether” background of the universe. Kelvin’s theory was immensely popular at the time and led to considerable efforts in the enumeration and classification of knots and links. Kelvin’s coauthor and colleague, the Scottish physicist Peter Guthrie Tait, published the first knot tables in 1876. He displayed and discussed the following links:
In this display, No. 15 shows the Borromean Rings, while No. 18 is an apparently different link that, however, shares the same characteristics: It consists of three closed curves that are pairwise not linked, whereas the whole diagram does not seem to come apart, it represents a nontrivial link.

Tait indeed claimed that the links No. 15 and No. 18 were not equivalent, apparently based on the assumption that any alternating diagram of a link (where along any string under- and over-crossings alternate) has a minimal number of crossings among all possible diagrams. This long-standing “Tait conjecture” was proved more than 100 years later, by Thistlethwaite, Kauffman, and Murasugi in 1987. (Tait’s examples No. 16 and 17 have only one component, so they are knots. All four examples fall into a larger family that has been described and studied as the “Turk’s head links.”)

In 1892, the geometer Hermann Brunn introduced a much more general family of objects that we now call Brunnian links: \( k \)-component links in which any subcollection of \( k - 1 \) of the components is trivial. Tait’s links No. 15 (the Borromean rings) and No. 18 are examples.

Back to the Borromean rings: Indeed they cannot be built from three perfect circles. The first proof for this appeared in 1987 in a long differential geometry paper by Michael F. Freedman and Richard Skora. Their beautiful geometric idea, “getting movies from spherical domes,” is very powerful: It solves the problem not only for the Borromean rings, but shows that any Brunnian link built from perfect circles is trivial. It can also be generalized to links formed by \( k \)-spheres in \( (2k + 1) \)-dimensional space. Our presentation is based on a short unpublished note “Circle links” by Ian Agol.

**Theorem 1.** If a link consists of disjoint perfect circles that are pairwise not linked, then the link is trivial.

**Proof.** Moving each of the circles just a little bit, we may assume that they lie in planes that are distinct, no two of the planes are parallel, and none of the planes spanned by one of the circles contains the center of a second circle. (This first preparatory step is not necessary, but it simplifies some later parts of the proof quite a bit.)

There are several different ways to define what it means that two disjoint circles in \( \mathbb{R}^3 \) are linked. Let us here use the following: Two circles are linked if one of them intersects (and not only touches) the disk spanned by the other one exactly once.

Let the circles be \( C, C' \subseteq \mathbb{R}^3 \), let \( D, D' \) be the flat disks they bound, and let \( H, H' \) be the planes they span. If \( C' \) intersects the disk \( D \) in one point, then this point lies both in \( D \subseteq H \) as well as on \( C' \subseteq D' \subseteq H' \), so in particular it lies in the intersection of the two planes \( H \) and \( H' \), which is a line, \( L := H \cap H' \). As this line lies in the plane \( H \) and contains a point in the interior of the disk \( D \), it intersects \( C \) in exactly two points. The circle \( C' \) intersects the plane \( H \) once in the interior of \( D \), so there has to be a second intersection point, which lies again on the line \( L \), but outside \( D \).