What is the most interesting formula involving elementary functions? In his beautiful article [2], whose exposition we closely follow, Jürgen Elstrodt nominates as a first candidate the partial fraction expansion of the cotangent function:

\[
\pi \cot \pi x = \frac{1}{x} + \sum_{n=1}^{\infty} \left( \frac{1}{x+n} + \frac{1}{x-n} \right) \quad (x \in \mathbb{R} \setminus \mathbb{Z}).
\]

This elegant formula was proved by Euler in §178 of his *Introductio in Analysin Infinitorum* from 1748 and it certainly counts among his finest achievements. We can also write it even more elegantly as

\[
\pi \cot \pi x = \lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{x+n}
\]

but one has to note that the evaluation of the sum \(\sum_{n \in \mathbb{Z}} \frac{1}{x+n}\) is a bit dangerous, since the sum is only conditionally convergent, so its value depends on the “right” order of summation.

We shall derive (1) by an argument of stunning simplicity which is attributed to Gustav Herglotz — the “Herglotz trick.” To get started, set

\[
f(x) := \pi \cot \pi x, \quad g(x) := \lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{x+n},
\]

and let us try to derive enough common properties of these functions to see in the end that they must coincide . . .

**A** The functions \(f\) and \(g\) are defined for all non-integral values and are continuous there.

For the cotangent function \(f(x) = \pi \cot \pi x = \pi \frac{\cos \pi x}{\sin \pi x}\), this is clear (see the figure). For \(g(x)\), we first use the identity \(\frac{1}{x+n} + \frac{1}{x-n} = -\frac{2x}{n^2 - x^2}\) to rewrite Euler’s formula as

\[
\pi \cot \pi x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2x}{n^2 - x^2}.
\]

Thus for **A** we have to prove that for every \(x \notin \mathbb{Z}\) the series

\[
\sum_{n=1}^{\infty} \frac{1}{n^2 - x^2}
\]

converges uniformly in a neighborhood of \(x\).
For this, we don’t get any problem with the first term, for \( n = 1 \), or with the terms with \( 2n - 1 \leq x^2 \), since there is only a finite number of them. On the other hand, for \( n \geq 2 \) and \( 2n - 1 > x^2 \), that is \( n^2 - x^2 > (n - 1)^2 > 0 \), the summands are bounded by

\[
0 < \frac{1}{n^2 - x^2} < \frac{1}{(n-1)^2},
\]

and this bound is not only true for \( x \) itself, but also for values in a neighborhood of \( x \). Finally the fact that \( \sum \frac{1}{(n-1)^2} \) converges (to \( \frac{\pi^2}{6} \), see page 53) provides the uniform convergence needed for the proof of (A).

(B) Both \( f \) and \( g \) are periodic of period 1, that is, \( f(x + 1) = f(x) \) and \( g(x + 1) = g(x) \) hold for all \( x \in \mathbb{R}\setminus\mathbb{Z} \).

Since the cotangent has period \( \pi \), we find that \( f \) has period 1 (see again the figure above). For \( g \) we argue as follows. Let

\[
g_N(x) := \sum_{n=-N}^{N} \frac{1}{x + n},
\]

then

\[
g_N(x + 1) = \sum_{n=-N}^{N} \frac{1}{x + 1 + n} = \sum_{n=-N+1}^{N+1} \frac{1}{x + n}
\]

\[
= g_{N-1}(x) + \frac{1}{x + N} + \frac{1}{x + N + 1}.
\]

Hence \( g(x + 1) = \lim_{N \to \infty} g_N(x + 1) = \lim_{N \to \infty} g_{N-1}(x) = g(x) \).

(C) Both \( f \) and \( g \) are odd functions, that is, we have \( f(-x) = -f(x) \) and \( g(-x) = -g(x) \) for all \( x \in \mathbb{R}\setminus\mathbb{Z} \).

The function \( f \) obviously has this property, and for \( g \) we just have to observe that \( g_N(-x) = -g_N(x) \).

The final two facts constitute the Herglotz trick: First we show that \( f \) and \( g \) satisfy the same functional equation, and secondly that \( h := f - g \) can be continuously extended to all of \( \mathbb{R} \).

(D) The two functions \( f \) and \( g \) satisfy the same functional equation: \( f(\frac{x}{2}) + f(\frac{x+1}{2}) = 2f(x) \) and \( g(\frac{x}{2}) + g(\frac{x+1}{2}) = 2g(x) \).

For \( f(x) \) this results from the addition theorems for the sine and cosine functions:

\[
f(\frac{x}{2}) + f(\frac{x+1}{2}) = \pi \left[ \cos \frac{\pi x}{2} \sin \frac{\pi x}{2} - \sin \frac{\pi x}{2} \cos \frac{\pi x}{2} \right]
\]

\[
= 2\pi \frac{\cos(\frac{\pi x}{2} + \frac{\pi x}{2})}{\sin(\frac{\pi x}{2} + \frac{\pi x}{2})} = 2f(x).
\]